A Theory of Machining of Fiber-Reinforced Materials

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A theory of the machining of fiber-reinforced materials is presented. The analysis is restricted to plane deformations of incompressible composites reinforced by strong parallel fibers. Complete deformation and stress fields, as well as estimates of the forces required to maintain continuous machining, are derived. The results apply to both elastic and plastic stress responses.

INTRODUCTION

RESEARCH INTO THE THEORY of the mechanics of machining has been mainly devoted to the mechanical cutting of isotropic metals [1].* In this paper we present an analysis of the machining problem as related to fiber-reinforced materials.

During the machining process a surface layer of material is removed by a wedge-shaped tool which is constrained to travel parallel to the surface of the workpiece. We are interested in analyzing the macroscopic behavior of the composite material and its reactions on the tool. The interaction between individual fibers and the surrounding matrix, and the detailed behavior of both constituents, are outside the scope of this work.

We consider materials composed of parallel strong fibers embedded in a weaker matrix. The fibers are initially aligned parallel to the direction of travel of the tool. We consider composites in which the bulk modulus of the material and its extensional modulus in the fiber direction are large in comparison with the shear moduli. Then, following the three-dimensional continuum model proposed by Mulhern, Rogers and Spencer [2], we idealize

* Numbers in square brackets indicate references at the end of the paper.

these conditions by treating the composite as incompressible, and inextensible along the fiber direction, and we treat the fibers as continuously distributed.

The general theory of large plane deformations of such "ideal" fiber-reinforced materials has been formulated in a recent paper by Pipkin and Rogers [3]. The results are not restricted to any particular form of stress response of the composite such as elasticity, plasticity or viscoelasticity.

In the present paper we apply this theory [3] to the problem of orthogonal cutting, in which the leading edge of the tool is perpendicular to the direction of relative motion between tool and workpiece. We assume both to be sufficiently broad, or to be so constrained, that plane strain conditions may be assumed. A complete statement of the problem and its assumptions is given in Section 2. In Section 3 we review those aspects of the general theory [3] that are relevant to this problem.

In the context of unreinforced, isotropic metals, there are two basic approaches to the analysis of machining. One is the thin-zone model (e.g. [4]-[7]) based on assuming that the plastic deformation is concentrated in a very narrow region in the chip, emanating from the tip of the tool. The other approach treats the plastic region as a thick zone, and includes the chip geometry suggested by Palmer and Oxley [8] in which the chip and workpiece are not in contact at the tool tip. In Section 4 we propose for the chip a deformation field analagous to this thick-zone model with rupture and separation occurring ahead of the tool. We obtain the complete stress solution, paying special attention to the thin layers of very high stress next to the bounding surfaces parallel to the fiber direction. These singular stress layers are a feature of the general theory, which also allows the presence both of stress concentration layers normal to the fiber directions and of points of stress concentration in regions with curved fibers. In Section 5 we consider the equilibrium of points at which these different effects coincide. The deformation and equilibrium of the finished piece—the remainder of the workpiece—are treated in Section 6.

The results of different material behaviors in shear are discussed in Section 7. In Section 8 we propose a tentative criterion for the separation of the chip from the rest of the slab in order to effect the continuous flow of the workpiece relative to the tool. An approximate expression is derived for the force required to maintain this flow. In the last section we present some conclusions to be drawn from the work, and we discuss possible alternative modes of deformation in the machining process.

STATEMENT OF THE PROBLEM

We consider a workpiece of thickness $h + H$ which is initially bounded by the planes $X_1 = 0, a$ and $X_2 = -H, h$ (Figure 1), and in which the

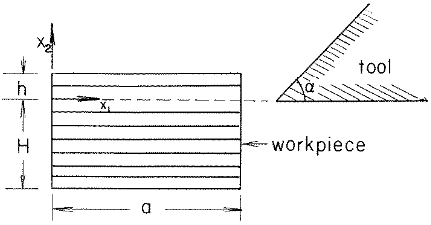

Figure 1. *Initial configuration.*

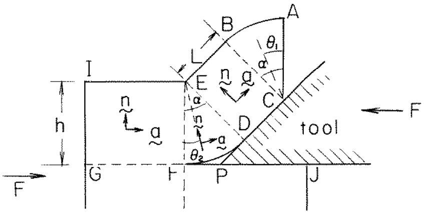

Figure 2. *Deformation of the chip.*

fibers are aligned in the X_1-direction. A surface layer of thickness h is removed by driving a rigid wedge-shaped tool, with rake angle $(\pi/2)$-α, into the slab with a relative motion in the X_1-direction.

The problem is to calculate the distributions of stress and displacement in the composite when the tool and workpiece are subjected to equal and opposite forces F per unit of length in the X_3-direction, and with the chip already formed. The actual development of the chip is not studied.

The material is homogeneous, incompressible, and inextensible in the fiber direction. The fibers, and thus the local inextensibility condition, are convected with the deformation. Plane strain conditions are assumed, so that we may neglect dependence on X_3. We also assume the body forces and inertial terms to be negligible.

Throughout the deformation, the end $X_1 = 0$ has zero or uniform displacement in the fiber direction. The remaining end $X_1 = a$ and lateral surfaces $X_2 = -H, h$ are traction-free. On the new surfaces created by the machining, displacement is obviously specified on the parts in contact with the tool, and zero tractions are assumed for the remainder. For convenience we assume the tool to be perfectly lubricated, implying zero shear tractions in the contact regions.

PREVIOUS THEORY

The theory of large, plane deformations of ideal composites has been formulated by Pipkin and Rogers [3] for arbitrary material behavior.

It was shown that the kinematical constraints in the theory require that initially parallel fibers remain parallel and that the distance between them is conserved throughout a deformation. The deformation is then locally a simple shear, with the amount of shear γ given by

$$d\mathbf{x} = (\mathbf{n} + \gamma \mathbf{a})\, dX_2 \,. \tag{3.1}$$

Here dX_2 is the length of a material line element initially lying parallel to the X_2-axis, and $d\mathbf{x}$ represents its deformed configuration; $\mathbf{a}$ and $\mathbf{n}$ denote unit vectors that are respectively tangential and normal to the fibers in the deformed state (Figure 2) and lie in the X_1-X_2 plane.

The analysis also showed that, under weak restrictions on the stress response of the composite, every kinematically admissible deformation gives rise to a stress field which automatically satisfies the equilibrium equations. Thus, in any mixed boundary-value problem, it only remains to verify that this stress field satisfies the prescribed traction boundary conditions as well.

If we make the weak assumption that the composite has reflectional symmetry in the plane of plane strain, then the stress tensor $\boldsymbol{\sigma}$ may be conveniently written in the dyadic form

$$\boldsymbol{\sigma} = -p(\mathbf{I} - \mathbf{aa}) + T\mathbf{aa} + S(\mathbf{an} + \mathbf{na}) + S_{33}\mathbf{kk}. \quad (3.2)$$

Here $\mathbf{I}$ is the unit tensor and $\mathbf{k}$ ($= \mathbf{a} \times \mathbf{n}$) is the unit vector in the X_3-direction. An alternative form of (3.2) is

$$\boldsymbol{\sigma} = \begin{bmatrix} T & S & 0 \\ S & -p & 0 \\ 0 & 0 & S_{33} - p \end{bmatrix} \quad (3.3)$$

with $\mathbf{a}$, $\mathbf{n}$ and $\mathbf{k}$ as the base vectors in the 1-, 2- and 3-directions, respectively. Thus, T is the total tension on elements normal to the fiber-direction and p is the total pressure on elements normal to the $\mathbf{n}$-direction.

The shearing stress S and the normal stress difference S_{33} depend only on the amount of shear γ or its history. Thus, for any particular composite, the dependence of S on γ can be determined from a simple shear test. This will normally be the only data required, since S_{33} is needed only for finding σ_{33}. In the present paper we need assume nothing about S_{33} except that it is independent of X_3.

T and p are stress reactions to the constraints of inextensibility and incompressibility, and are determined from the equilibrium equations. For composites with parallel fibers these equations take the form [3]

$$\mathbf{a} \cdot \tilde{\nabla} T = 2\varkappa S - \mathbf{n} \cdot \tilde{\nabla} S \quad (3.4)$$

$$\mathbf{n} \cdot \tilde{\nabla} p = \varkappa(p + T) + \mathbf{a} \cdot \tilde{\nabla} S \quad (3.5)$$

and $\partial p / \partial X_3 = 0$. Here $\varkappa$ denotes the fiber curvature.

SOLUTION FOR THE CHIP REGION

The displacement field that we propose for the chip region is analogous to the thick-zone model for isotropic metals which has rupture and separation of the material ahead of the tool tip [8]. It is shown in Figure 2 and consists only of straight segments and sectors of circles (fans). In each fan the n-lines are all radial lines, and the fibers are all arcs of concentric circles. The thickness of the chip layer is unchanged, with constant value h. Thus the fibers all remain parallel and at the same distance apart, satisfying the conditions of admissibility. The proposed deformation also satisfies fiber-inextensibility and the displacement boundary conditions stated in Section 2.

Within the fans we define local polar coordinates (r_1, θ_1) and (r_2, θ_2) where the radii are measured from the fan centers, and the angles are positive, measured from the vertical (Figure 2). Then, from previous theory [9] or by applying (3.1) directly, the amount of shear in the chip layer is

$$\gamma = (\theta_1, \alpha, \theta_2, 0) \text{ in } (ABC, BCDE, DEF, EFGI). \tag{4.1}$$

In the fan regions the directional derivatives take the forms

$$(\underset{\sim}{\mathbf{a}} \cdot \nabla, \underset{\sim}{\mathbf{n}} \cdot \nabla) = \begin{cases} \left(-\dfrac{1}{r_1} \dfrac{\partial}{\partial \theta_1}, \dfrac{\partial}{\partial r_1} \right) & \text{in } ABC \\ \left(\dfrac{1}{r_2} \dfrac{\partial}{\partial \theta_2}, -\dfrac{\partial}{\partial r_2} \right) & \text{in } DEF. \end{cases} \tag{4.2}$$

Then from (3.4), (3.5), (4.1) and (4.2) we obtain

$$T = T(r, \theta_0) + \int_{\theta_0}^{\theta} \left(2S + r \frac{\partial S}{\partial r} \right) d\theta' \tag{4.3}$$

and

$$p = \frac{r_0}{r} p(r_0, \theta) + \frac{1}{r} \int_r^{r_0} \left(T + \frac{\partial S}{\partial \theta} \right) dr'. \tag{4.4}$$

Here (r, θ) represents (r_1, θ_1) in ABC and (r_2, θ_2) in DEF; $T(r, \theta_0)$ is the tension, in the fiber direction, acting on the normal line $\theta = \theta_0$, and $p(r_0, \theta)$ is the pressure acting on the fiber-line $r = r_0$.

In a straight segment, $\varkappa = 0$ and the directional derivatives are

$$\underset{\sim}{\mathbf{a}} \cdot \nabla = -\partial/\partial s, \quad \underset{\sim}{\mathbf{n}} \cdot \nabla = \partial/\partial \xi \tag{4.5}$$

where s is the arc length measured along a fiber in the $-\mathbf{a}$ direction, and ξ is the perpendicular distance from the innermost fiber $(X_2 = 0)$. Then (3.4) and (3.5) simply give

$$T = T(\xi, s_0) + \int_{s_0}^{s} \frac{\partial S}{\partial \xi} ds' \tag{4.6}$$

and

$$p = p(\xi_0, s) - \int_{\xi_0}^{\xi} \frac{\partial S}{\partial s} d\xi'. \tag{4.7}$$

In general, the condition of zero traction on a surface with normal $\boldsymbol{\nu}$ implies that the stress vector $\boldsymbol{\sigma\nu}$ is zero. In the present problem all the free surfaces are perpendicular to either $\mathbf{a}$ or $\mathbf{n}$. Hence the traction boundary conditions on the chip (Section 2) imply

$$T = S = 0 \quad \text{on} \quad AC \ (\boldsymbol{\nu} = \mathbf{a}) \tag{4.8}$$

$$p = S = 0 \quad \text{on} \quad AI \text{ and } DF \ (\boldsymbol{\nu} = \mathbf{n}, -\mathbf{n}). \tag{4.9}$$

The details of the complete stress solution can now be presented. From (4.1) we see that $\partial S/\partial r$ and $\partial S/\partial \xi$ vanish everywhere inside the chip region, for $0 < \xi < h$. Using (4.8), with (4.3) and (4.6) gives

$$T = 2\int_0^\theta S(\eta)d\eta, \quad 0 < \xi < h \tag{4.10}$$

where θ is equal to γ given in (4.1).

A different situation exists in the bounding fiber layers $AI(\xi = h)$ and $CG(\xi = 0)$. There T is *not* given by (4.10). The condition of zero shear traction on AE and CF implies that S changes discontinuously from a non-zero value on the interior of the chip to zero on the exterior of the chip. The terms $\partial S/\partial r$ and $\partial S/\partial \xi$ in (4.3) and (4.6) then contribute a singular term in T at $\xi = 0$ and $\xi = h$. These two layers of stress singularities each carry a finite load. Hence

$$T = \begin{cases} T_h^* \ \delta(\xi - h) & \text{in } AI \\ \\ T_0^* \ \delta(\xi) & \text{in } CG \end{cases} \tag{4.11}$$

where $\delta(\xi)$ is the Dirac delta or unit impulse function. T_0^* and T_h^* are the finite loads given by (4.3) and (4.6) as

$$T_0^* = \begin{cases} -h\int_0^\alpha S(\eta)d\eta + sS(\alpha) & \text{in } CD \\ LS(\alpha) - h\int_0^{\theta_2} S(\eta)d\eta & \text{in } DF \\ LS(\alpha) & \text{in } FG \end{cases} \tag{4.12}$$

and

$$T_h^* = \begin{cases} -h\int_0^{\theta_1} S(\eta)d\eta & \text{in } AB \\ -h\int_0^\alpha S(\eta)d\eta - sS(\alpha) & \text{in } BE \\ -F_1 - LS(\alpha) & \text{in } EI \end{cases} \tag{4.13}$$

where F_1 is the portion of F applied to the chip layer, L is the length of chip in contact with the tool, and s is the distance from BC. The value of T_h^* in EI is given by equilibrium of $EFGI$ treated as a free body; we see from (4.13) that T_h^* is discontinuous at E. Similarly we note from (4.12a) that T_0^* is non-zero at C; thus, in order to satisfy the boundary condition (4.8), T_0^* is discontinuous at C. E and C are singular points, which are considered in detail in the next section.

The singular behavior of T in the surface layers, as exhibited in (4.11), is a feature of the general theory [3] and has been discussed elsewhere [3, 9]. In a real composite, the extensional modulus in the fiber direction is finite, but large compared with the shear moduli. We then interpret the layers of stress concentration to be thin boundary layers of high stress. These thin layers have been found in simpler problems [10] involving infinitesimal deformations of transversely isotropic materials.

Apart from σ_{33}, the stress solution is completed by determining p. Using (4.10) to (4.13) for T in (4.4) and (4.7), and integrating, gives

$$p = \begin{cases} \dfrac{1}{\xi}\left[(h-\xi)S'(\theta_1) + \{2(h-\xi) - hU(h-\xi)\}\displaystyle\int_0^{\theta_1} S(\eta)d\eta \right] & \text{in } ABC \\ 0 \text{ in } BCDE \quad \text{and} \quad EFGI & \\ \dfrac{1}{h-\xi}\left[\xi S'(\theta_2) + LS(\alpha)U(\xi) + \{2\xi - hU(\xi)\}\displaystyle\int_0^{\theta_0} S(\eta)d\eta \right] & \text{in } DEF \end{cases} \tag{4.14}$$

where $S'(\theta) \equiv dS/d\theta$ and $U(\xi)$ is the Heaviside, or Unit, step function defined as

$$U(\xi) = \begin{cases} 0, & \xi \leqq 0 \\ 1, & \xi > 0\,. \end{cases} \tag{4.15}$$

We note that p is discontinuous across the two boundary fiber layers at AB and DF, becoming zero on the outer surface thus satisfying the boundary conditions (4.9).

EQUILIBRIUM OF SINGULAR POINTS

The **a**-lines and **n**-lines are two orthogonal families of characteristics for the hyperbolic differential equations (3.4) and (3.5). If two characteristics of the same family meet, we term the point of intersection "singular". C and E are thus singular points. At such a point the differential equations must be replaced by integrated forms. In our case, (3.4) and (3.5) are equilibrium equations, and the integrated forms reduce to elementary statics.

For equilibrium of the point C it is convenient instead to consider ABC as a free body (Figure 3a). Equilibrium in the direction normal to BC requires

$$T_0^*(0) = -\,T_h^*(0) - hT(\alpha) \tag{5.1}$$

where $T_h^*(0)$ is given by (4.13b) with $s = 0$, and $T(\alpha)$ by (4.10) with $\theta = \alpha$. As noted previously (Section 4) this means that T_0^* must suffer a discontinuous change at C from zero on the outer surface ($s = 0^-$) to $T_0^*(0)$

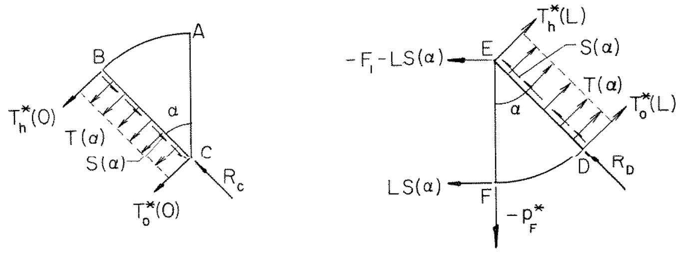

***Figure 3.** Equilibrium of singular points C and E.*

on the interior ($s = 0^+$).

Equilibrium in the direction of BC demands that the tool provide a singular normal pressure at C with resultant R_C given by

$$R_C = hS(\alpha). \tag{5.2}$$

The point E is conveniently treated by considering DEF as a free body (Figure 3b). The resultant of the normal forces on DE is zero, as seen from (4.10)-(4.13) or by considering the equilibrium of $BCDE$. Hence, in order to maintain vertical equilibrium (i.e., in the direction of EF) a resultant pressure must be applied at E, or a resultant *tension* must be applied on DF. The pressure applied at E must be zero if the surface is assumed free of traction during the machining process. The tool cannot provide a negative pressure. Thus the equilibrating tensile force can act on DF only through a singular tensile force $-p_F^*$ acting at F, the point of separation.

At the same time there is no reason, a priori, why the normal line DE should not carry a singular stress supplied by a finite *compressive* load R_D exerted by the tool at D. Equilibrium of DEF then requires

$$R_D = F_1 \operatorname{cosec} \alpha - hS(\alpha) \tag{5.3}$$

and

$$p_F^* = -F_1 \cot \alpha. \tag{5.4}$$

At this stage we note that it is known [3, 9] that normal lines can and sometimes must carry finite loads. However, all the previous examples show the singular pressures as arising from the term $\partial S/\partial\theta$ in (4.4) when the shear stress is discontinuous across some normal line. The present case is an example of singular normal lines occurring even when the shear stress is continuous across all normal lines.

The complete set of forces acting on the chip layer is now known and shown in Figure 4. Here the reactions R_G^+ and R_I are simply $-T_0^*$ and $-T_I^*$ as given in (4.12c) and (4.13c). The superscript on R_G is used to denote that portion of the total reaction R_G which is due to the shearing of the chip. In

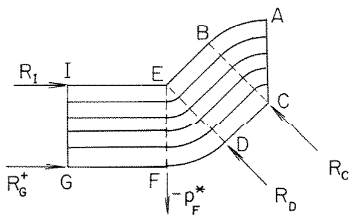

***Figure 4.** External loading of the chip.*

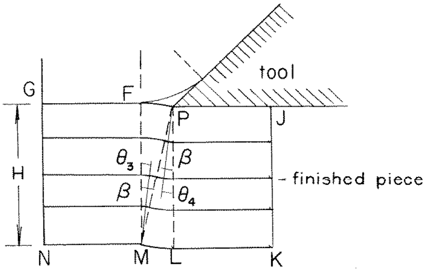

***Figure 5.** Deformation of the finished piece.*

the next section we obtain an additional contribution R_G^- due to the shearing of the finished piece.

Finally we note that the condition $R_D \geq 0$, when inserted in (5.3), shows that the range of validity of our entire solution is restricted. The end loading F_1 must satisfy

$$F_1 \geq hS(\alpha) \sin \alpha. \tag{5.5}$$

EQUILIBRIUM OF THE FINISHED PIECE

In Figure 2 we depict the finished piece as having suffered no deformation. However, (5.4) shows that the chip exerts a tensile load on it at F. According to our idealized theory, this stress singularity would be transmitted without attenuation along the normal FM. If the lower surface of the workpiece were rigidly held, this tensile load would be automatically equilibrated. Then the solution for the entire problem would be complete.

If, however, this lower surface were free, or subjected only to pressure, then the tension $-p_F^*$ can be equilibrated only by an equal pressure exerted by the tool. This would shear the finished piece in the vicinity of FM and the tool tip P, suggesting that the deformation will be as shown in Figure 5, particularly if the underside of the tool is horizontal. PMF and PML are both fans of angle β, which is determined in terms of α and h/H through

$$\cos(\alpha + \beta) = \cos\alpha - \frac{h}{H}(1 - \cos\alpha). \tag{6.1}$$

The deformation is evidently kinematically admissible. The amount of shear is

$$\gamma = (-\theta_3, -\theta_4) \text{ in } (PMF, PML) \tag{6.2}$$

and zero elsewhere.

Although deformation of the finished piece might normally be considered unusual, its importance has been exaggerated in Figure 5. In practice, the ratio $h/H << 1$, so that (6.1) gives

$$\beta \sim \frac{h}{H} \frac{1 - \cos \alpha}{\sin \alpha}, \qquad (6.3)$$

and the downward deflection d of the finished piece as it passes beneath the tool is given by

$$\frac{d}{H} = 1 - \cos \beta = 0\left(\frac{h^2}{H^2}\right). \qquad (6.4)$$

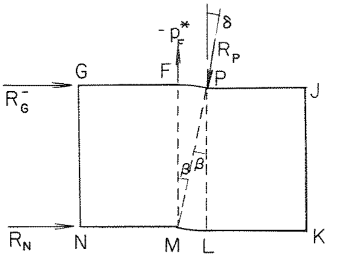

Figure 6. *Equilibrium of the finished piece.*

Thus the "wrinkle" ML would not usually be observable.

The stress analysis is entirely similar to that described in Sections 4 and 5 for the chip, and details will be omitted. Again the stresses in the bounding fiber layers are singular, and singular pressures are required to equilibrate the singular points P and M. In the straight segments, $p = T = 0$. The point forces acting on the finished piece are shown in Figure 6. We neglect any pressure distribution which might be exerted on KL, since it would be equilibrated by an equal and opposite pressure distribution exerted by the tool.

Since the leading edge of the tool is at P, where no unique normal exists, the reaction R_p there can have both vertical and horizontal components non-zero, as shown. However, if the tool tip is perfectly lubricated, R_p must lie between the two limiting normals PM and PL. Thus,

$$0 \leqq \delta \leqq \beta. \qquad (6.5)$$

Equilibrium of the finished piece (Figure 6) requires

$$R_p = -p_F^* \sec \delta \qquad (6.6)$$

$$R_G^- = p_F^* (\sin \beta - \tan \delta \cos \beta) \qquad (6.7)$$

$$R_N = -p_F^* \{\sin \beta + (1 - \cos \beta) \tan \delta\} \qquad (6.8)$$

where p_F^* is given by (5.4) in terms of F_1, the portion of F acting on the chip layer. Note that, in practice, with $h << H$, the loads R_G^- and R_N applied to the driven end are negligible compared with F.

PLASTIC BEHAVIOR

The general theory of large plane deformations of composites [3], on which the preceding analysis is based, is valid for elastic and plastic material behaviors. For an elastic material, the shearing stress S depends only on γ; hence the results obtained in the previous sections are valid without change for elastic stress response, since $S(\gamma)$ has been treated as a function of γ.

For plastic behavior the solution presented is valid only if the state of stress in the workpiece is monotone non-decreasing in time. Otherwise the shear stress S is not a single-valued function of γ, having a different value during unloading of an element; in this case the solution must be modified slightly. For the chip region unloading is possible only in the fan ABC. The state of stress there depends on the manner in which the fan was formed during the early stages of separation of the workpiece. If we assume that no unloading occurred during the incipient flow, then the chip solution we have obtained is valid without change for plastic as well as elastic behavior.

For the finished piece, the effect of plasticity on the solution is to cause S to vary rapidly (discontinuously for rigid/plastic behavior) from $S(\beta)$ to zero across PM, since unloading has occurred in the fan PML, however the chip was formed. In this case $T = p = 0$ in PML, so that this fan is now a "dead" region of zero stress. The remaining part of the solution would proceed as before but with the new condition that $T = 0$ on PM.

A CRITERION FOR CONTINUOUS MACHINING

The total force F applied to the left end of the slab can be expressed directly in terms of p_F^*. From (5.4), (6.7) and (6.8) we have

$$F = -p_F^* (\tan \alpha + \tan \delta). \tag{8.1}$$

The configuration we have proposed is in static equilibrium for all end forces F; any increase in F can be equilibrated with a corresponding increase in $-p_F^*$, so that the workpiece need not move relative to the tool. However, this result is based on an assumption of infinite ultimate strength for the composite, enabling the material to withstand any tensile stress without rupturing. A real material has only limited strength in any direction. In our case, rupture of the workpiece at the point of separation F would allow relative motion to occur.

Thus (8.1) furnishes the basis for a possible criterion for determining the force F_c required to maintain continuous machining. Equations (4.14) and (5.4) show that the maximum tensile load normal to the fibers is $-p_F^*$ concentrated at F. In fact, $-p_F^*$ is the force needed to equilibrate the resultant of the forces converging on the singular point E; it is transmitted to F along the normal line EF. In a real material, this singular response along EF should be interpreted as a thin layer of high stresses. Just as with the singular boundary layers $\xi = 0$ and $\xi = h$, a much more sophisticated theory is required in order to determine the details of such a stress distribution. We do not attempt such an analysis here.

Instead, we note that the neighborhood of EF is a region of small deformation, so that a linear stress analysis should be relevant. Such an analysis has already been made [10] for the simpler, but similar, problem of a line-force loading of a transversely isotropic elastic half-space. It shows that the stress is indeed concentrated within a narrow channel below the surface. The maximum stress at any depth occurs directly below the point of application. For a tensile loading $-p_F^*$ (per unit of length in the X_3-direction) the compressive stress p is given by [10]

$$p \sim p_F^* / \epsilon \pi h \qquad (8.2)$$

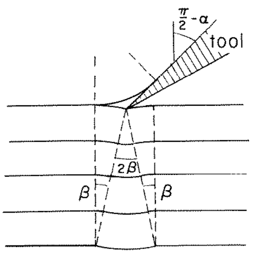

Figure 7. *Alternative deformation of the finished piece.*

at a depth h. Here ϵ is a small dimensionless parameter given by

$$\epsilon^2 = 2(1-\nu)\,\mu/E\,. \qquad (8.3)$$

Thus, in (8.2), higher order terms have been neglected. In (8.3), E is the extensional modulus in the transverse direction, μ is the modulus of shear along the fibers, and $\nu(\cong 1)$ is the Poisson's ratio of contraction to extension in the plane of transverse isotropy.

Hence, in our case we can interpret equations (8.1) and (8.2) to show that a reasonable, but tentative, estimate of F_c is provided by

$$F_c = \epsilon \pi h \; T_U \,(\tan\alpha + \tan\delta)\,, \qquad (8.4)$$

where T_U is the composite's ultimate tensile strength normal to the fiber direction.

CONCLUSION

In this study we have presented a theoretical analysis of machining, along the grain, of materials reinforced with strong fibers. We have shown that the proposed deformation and the resulting stress field are compatible with specified boundary conditions of the problem. The results can be interpreted for both elastic and plastic material responses.

No claim is made that the mode of deformation described is the only one possible, even within the constraints of the theory. In particular, there is no reason, a priori, to exclude a displacement field analogous to the thin zone model for isotropic metals.

The possible deformations of the finished piece appear to depend on the tool shape. Having assumed the underside of the tool to be horizontal near the tip P, the deformation of Figure 5 was proposed. If, however, the included angle at the tip were less than $\alpha - \beta$ (while maintaining the same rake angle $(\pi/2) - \alpha$), the finished piece could deform as shown in Figure 7. The stress solution here could be obtained by straightforward analysis similar to that of Sections 4 and 5.

These uncertainties might be resolved by analyzing the more complex problem of the incipient development of the chip. In the context of isotropic metal cutting, for example, Hill [11] suggests that the ultimate steady-state flow may be determined by the set of initial conditions. On the other hand, experimental observations would also help, if only to determine if and when separation occurs ahead of the tool tip.

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