

Stress Channelling in Transversely Isotropic Elastic Composites

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1. Introduction

The theory of plane deformations of ideal fiber-reinforced composites [1] is a simplified theory of the behavior of materials composed of strong fibers embedded in a weaker matrix. The idealized theory is based on the fictions that the fibers are continuously distributed and inextensible, and that the composite is incompressible in bulk. These idealizations make it easy to solve problems that would otherwise be regarded as extremely difficult [2–6].

The classical theory of infinitesimal elastic deformations of transversely isotropic materials covers some of the same ground. Since the theory of ideal fiber-reinforced composites includes infinitesimal elastic response as a special case, the validity of the general theory would certainly be doubtful if it did not agree with classical elasticity theory when the latter is applicable.

This raises some misgivings, because the predictions of the idealized theory are often quite unlike anything one may have encountered in classical elasticity theory. The explanation of the apparent discrepancy is that intuition about classical elasticity theory is usually based on the theory of *isotropic* materials, and even the methods of approximation [7] used for anisotropic materials are based on the gratuitous assumption that what works well for isotropic materials will still be valid for strongly anisotropic materials.

There are two predictions of the idealized theory that *appear* to be most fundamentally at variance with classical elasticity theory. First, in most problems there are infinitesimally thin layers of infinite tensile stress, either along a fiber or in a normal line, perpendicular to the fibers. Second, in disagreement with Saint-Venant's principle, conditions at one point on a fiber or normal line can strongly affect conditions far away along that fiber or normal line.

One object of the present paper is to show that these results are in no way contradictory to classical elasticity theory; they merely draw sharp attention to results, available by classical methods, that previously have been ignored. We examine some simple, exact elastic solutions in the limit of small extensibility and compressibility. We show that the results of the idealized theory do indeed represent limiting cases of the results given by the more complicated classical methods. Singular fibers and normal lines are found to represent limiting cases of very thin layers of very high tensile stress concentration. Unattenuated propagation of disturbances along fibers or normal lines is found to represent extremely slow decay along such lines.

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A second objective is to obtain order of magnitude estimates of the thicknesses of stress concentration layers, the tensile stresses in them, and their lengths, in order to be able to give a better interpretation to results of the idealized theory when this theory predicts zero thickness, infinite stress, and arbitrarily large lengths. We summarize these estimates in Section 9.

A third objective is to lay the groundwork, by means of specific examples, for a boundary-layer (singular perturbation) theory that can use the idealized theory as a starting point for more refined stress analyses. However, very little is said about boundary-layer theory in the present paper.

In Section 2 we review the subject of plane deformations of transversely isotropic elastic materials, and in Section 3 we show that the material properties relevant to plane deformations can be lumped into two dimensionless parameters ϵ_l and ϵ_c which are small when the extensibility and compressibility of the material are small in comparison to its shear compliances. In Section 4 we review the analogous idealized theory, with zero extensibility and compressibility.

In Sections 5 to 8 we compare a number of solutions from elasticity theory with the corresponding solutions from the idealized theory. All of these solutions are trivial; the object is only to examine the limiting behavior of the elastic solutions. The particular solutions considered are the fundamental solutions for a half-space, with sinusoidal or point loading on the boundary.

2. Plane Elastic Deformations of Transversely Isotropic Materials

We consider infinitesimal elastic deformations of materials composed of straight, parallel fibers, distributed uniformly in a matrix of more compliant material. For macroscopic analyses we can treat such composites as homogeneous, and transversely isotropic about the fiber direction. If we take the x_3 -direction to be the preferred direction, the stress-strain relations have the forms

$$\left. \begin{aligned} \epsilon_{33} &= \frac{1}{E} \sigma_{33} - \frac{\nu}{E} \sigma_{\alpha\alpha}, & \epsilon_{3\alpha} &= \frac{1}{2G} \sigma_{3\alpha}, \\ \epsilon_{\alpha\beta} &= \frac{1+\nu'}{E'} \sigma_{\alpha\beta} - \left(\frac{\nu'}{E'} \sigma_{\gamma\gamma} + \frac{\nu}{E} \sigma_{33} \right) \delta_{\alpha\beta}. \end{aligned} \right\} \quad (2.1)$$

Here Greek subscripts have the range 1, 2, and a repeated index indicates summation over that range. It can be shown that positive definiteness of the strain energy density is equivalent to the inequalities

$$G > 0, \quad E > 0, \quad E' > 0, \quad 1 - 2\nu^2 \frac{E'}{E} > \nu' > -1. \quad (2.2)$$

We consider cases in which the constraints of inextensibility in the fiber direction, $\epsilon_{33} = 0$, and bulk incompressibility, $\epsilon_{\alpha\alpha} = 0$ (given $\epsilon_{33} = 0$), might be appropriate idealizations. From (2.1) we see that $\epsilon_{33} = 0$ for all stresses if $E = \infty$, and in that case $\epsilon_{\alpha\alpha} = 0$ for all stresses if $\nu' = 1$. What we actually assume is that the extensional modulus E in the fiber direction is much larger than the shear modulus G and the

transverse extensional modulus E' , and that ν' is close to unity, as limited by (2.2d):

$$\frac{G}{E} \ll 1, \quad \frac{E'}{E} \ll 1, \quad 1 - \nu' \ll 1. \tag{2.3}$$

The contraction ratio ν is limited only by the assumption that it is not extremely large.

We consider plane deformations in the x, z plane ($x = x_1, y = x_2, z = x_3$). For such deformations, $\epsilon_{yx}, \epsilon_{yy}$ and ϵ_{yz} are zero. Hence, σ_{yx} and σ_{yz} are zero, and

$$\sigma_{yy} = \nu' \sigma_{xx} + \nu \frac{E'}{E} \sigma_{zz}. \tag{2.4}$$

Then

$$\epsilon_{zz} = A \sigma_{zz} - B' \sigma_{xx}, \quad \epsilon_{xx} = -B' \sigma_{zz} + C \sigma_{xx}, \tag{2.5}$$

and

$$\epsilon_{xz} = \frac{\sigma_{xz}}{2G}. \tag{2.6}$$

In (2.5), A, B' , and C are defined by

$$A = \frac{1}{E} \left(1 - \nu^2 \frac{E'}{E} \right), \quad B' = \frac{\nu}{E} (1 + \nu'), \quad C = \frac{1 - (\nu')^2}{E'}. \tag{2.7}$$

The equilibrium equations are satisfied if the stress is given in terms of Airy's stress function χ by

$$\sigma_{zz} = \chi_{,xx}, \quad \sigma_{xx} = \chi_{,zz}, \quad \sigma_{xz} = -\chi_{,xz}. \tag{2.8}$$

The strain compatibility equations yield one non-trivial relation,

$$\epsilon_{zz,xx} + \epsilon_{xx,zz} - 2\epsilon_{xz,xz}. \tag{2.9}$$

By using (2.5), (2.6) and (2.8) in (2.9), we obtain

$$A \chi_{,xxxx} + B \chi_{,xxxx} + C \chi_{,zzzz} = 0, \tag{2.10}$$

where

$$B = \left(\frac{1}{G} \right) - 2 B'. \tag{2.11}$$

We note that if (2.3) holds, then $B \cong 1/G$ and $A C/B^2 \ll 1$.

3. The Fundamental Parameters for Plane Deformations

It is convenient to write (2.10) in the factored form

$$\left(\epsilon_t^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \left(\frac{\partial^2}{\partial x^2} + \epsilon_c^2 \frac{\partial^2}{\partial z^2} \right) \chi = 0. \tag{3.1}$$

Here $1/\varepsilon_t^2$ and ε_c^2 are the larger and smaller roots λ of the equation

$$A \lambda^2 - B \lambda + C = 0. \tag{3.2}$$

Since $A C/B^2 \ll 1$ when either E is large in comparison to G or ν' is close to unity, the roots λ of (3.2) are approximately B/A and C/B . Thus, with the assumptions (2.3), we have

$$\varepsilon_t^2 \cong \frac{A}{B} \cong \frac{G}{E} \tag{3.3}$$

and

$$\varepsilon_c^2 \cong \frac{C}{B} \cong (1 - \nu') \frac{2G}{E}. \tag{3.4}$$

Of course, it is trivial to obtain exact expressions for these parameters, but the approximate expressions are more revealing.

We consider both ε_t and ε_c to be small parameters. From (2.2d) we find that we cannot take the limit $\varepsilon_c \rightarrow 0$ when ε_t is finite; indeed, since the product of the roots of (3.2) is C/A , then with (2.2) we obtain

$$\frac{\varepsilon_c^2}{\varepsilon_t^2} = \frac{C}{A} = \frac{E}{E'} \frac{1 - (\nu')^2}{1 - \nu^2(E'/E)} > \frac{2(1 + \nu')\nu^2}{1 - \nu^2(E'/E)} > 2(1 + \nu')\nu^2. \tag{3.5}$$

However, we can consider the limit $\varepsilon_c \rightarrow 0$ whenever $\varepsilon_t \rightarrow 0$ as well, with $\varepsilon_c > 2\nu\varepsilon_t$ as a rough bound. This will cause no difficulty.

From equation (3.1), we observe that the interchange of x with z is equivalent to the interchange of ε_t with ε_c . Thus, from any solution χ , we can generate the solution of the associated problem having the fiber and normal directions interchanged merely by interchanging ε_t with ε_c . In this sense, there is a complete duality between fibers and normal lines.

It will be recognized that for the isotropic case, $\varepsilon_t = \varepsilon_c = 1$ and (3.1) becomes the usual biharmonic equation. However, when ε_c and ε_t are very small, we are faced with a singular perturbation problem. Setting $\varepsilon_c = \varepsilon_t = 0$ in the equation gives a hyperbolic equation whose solutions ordinarily cannot satisfy the boundary conditions of an elliptic problem. The idealized theory gives a prescription for dealing with this anomalous situation.

4. The Idealized Theory

In the idealized theory, it is assumed at the outset that the constraint conditions,

$$\varepsilon_{33} = 0 \quad \text{and} \quad \varepsilon_{\alpha\alpha} = 0, \tag{4.1}$$

are valid. In plane deformations with $\varepsilon_{22} = 0$, it follows that $\varepsilon_{11} = 0$ as well. Hence, the displacements $u, v, w(u_1, u_2, u_3)$ have the form

$$u = u(z), \quad v = 0, \quad w = w(x). \tag{4.2}$$

The shearing stress component σ_{xz} is given at interior points by

$$\sigma_{xz} = 2G \varepsilon_{xz} = G [u'(z) + w'(x)]. \tag{4.3}$$

However, as we shall see, it is compatible with the equilibrium equations for the shearing stress to be discontinuous across a fiber $x = \text{constant}$ or a normal line $z = \text{constant}$. Consequently, on a *boundary* that is a fiber or normal line, prescribed shearing tractions need not agree with (4.3).

The stress equations of equilibrium yield

$$\sigma_{xx} = - \int^x \frac{\partial \sigma_{xz}}{\partial z} dx \quad (4.4)$$

and

$$\sigma_{zz} = - \int^z \frac{\partial \sigma_{xz}}{\partial x} dz. \quad (4.5)$$

The stress components σ_{xx} and σ_{zz} are reactions to the constraint conditions $\epsilon_{xx} = 0$ and $\epsilon_{zz} = 0$. Consequently, they do not satisfy constitutive equations, but are determined solely from the equilibrium equations (4.4) and (4.5), and the boundary conditions.

In Sections 5 to 8 we give a number of examples to show that this simple theory does yield results compatible with the elastic theory outlined in Sections 2 and 3 in the limit as ϵ_f and ϵ_c approach zero.

5. Singular Fibers and Normal Lines

As the first example we consider a half-space $x \geq 0$, with fibers parallel to the boundary, which is loaded by a sinusoidally varying tangential traction:

$$\sigma_{xz} = \sigma_0 \cos \frac{z}{L}, \quad \sigma_{xx} = 0 \quad (x = 0). \quad (5.1)$$

The stress at $x = \infty$ is zero.

Within the idealized theory, the displacements u and w are zero, and thus $\sigma_{xz} = 0$ at interior points. From the equilibrium equations (4.4) and (4.5) and the boundary condition (5.1b), it follows that at interior points,

$$\sigma_{xz} = 0 \quad \text{and} \quad \sigma_{zz} = f(x) \quad (x > 0). \quad (5.2)$$

It is a failing of the idealized theory that the tensile stress σ_{zz} in the fiber direction is indeterminate to the extent of a tension $f(x)$ which is constant along each fiber. This indeterminacy is not resolved by boundary conditions because here, no fiber crosses an external boundary. It is natural to specify that $f(x) = 0$, and we shall find that this is the choice directed by the elastic solution.

With $\sigma_{xz} = 0$ for $x > 0$, it follows from (5.1a) that σ_{xz} is discontinuous at $x = 0$, and its normal derivative is

$$\frac{\partial \sigma_{xz}}{\partial x} = -\sigma_0 \cos \frac{z}{L} \delta(x), \quad (5.3)$$

where $\delta(x)$ is the Dirac delta. Then (4.5) yields

$$\sigma_{zz} = \sigma_0 L \sin \frac{z}{L} \delta(x), \quad (5.4)$$

apart from a constant tension which we arbitrarily set equal to zero. Thus, the stress in the boundary fiber $x = 0$ is infinite, with a finite resultant tensile force $\sigma_0 L \sin(z/L)$.

We now examine the elastic solution of the same problem (Sections 2 and 3). It is easily found that the stress function is

$$\chi = \sigma_0 L^2 \varepsilon_t (1 - \varepsilon_t \varepsilon_c)^{-1} \sin \frac{z}{L} \left[\exp \frac{-x}{L \varepsilon_t} - \exp \frac{-x \varepsilon_c}{L} \right]. \tag{5.5}$$

Thus, for ε_t and ε_c small, the stress components are given by

$$\left. \begin{aligned} \sigma_{xx} &= 0(\varepsilon_t), & \sigma_{xz} &= \sigma_0 \cos \frac{z}{L} \exp \frac{-x}{L \varepsilon_t} + 0(\varepsilon_c \varepsilon_t), \\ \sigma_{zz} &= \sigma_0 L \sin \frac{z}{L} (L \varepsilon_t)^{-1} \exp \frac{-x}{L \varepsilon_t} + 0(\varepsilon_t). \end{aligned} \right\} \tag{5.6}$$

We find that the strange results of the idealized theory are, indeed, limits of elasticity theory for ε_t and ε_c approaching zero. First, (5.6a) gives $\sigma_{xx} = 0$ when $\varepsilon_t = 0$. Next, from (5.6b) we find that although the shearing stress is not discontinuous at the boundary, it decays from the prescribed boundary value to a negligible value within a boundary layer whose thickness is of the order $L \varepsilon_t$, which approaches zero with ε_t . Finally, we find that the infinite tensile stress (5.4) agrees exactly with (5.6c) if we interpret the delta in (5.4) as the limit for $\varepsilon_t \rightarrow 0$ of

$$(L \varepsilon_t)^{-1} \exp \frac{-x}{L \varepsilon_t}. \tag{5.7}$$

a function concentrated near $x = 0$, and very large there, whose integral over $x \geq 0$ is equal to unity.

In this example we find that a singular fiber should be interpreted as a thin layer whose thickness is $0(\varepsilon_t)$ in comparison to its characteristic length L . The tensile stress σ_{zz} in such a layer is of the order of the resultant force $\sigma_0 L \sin(z/L)$ predicted by the idealized theory, divided by the thickness $\varepsilon_t L$ of the layer, and thus of order $(\sigma_0/\varepsilon_t) \sin(z/L)$. We see that this rough estimate is accidentally equal to the value of σ_{zz} given by (5.6c) at $x = 0$.

For fibers orthogonal, instead of parallel, to the boundary, the idealized solution is unaffected, and the elastic solution is obtained from (5.6) by interchanging ε_t with ε_c . Thus, the line $x = 0$ is now a singular *normal line* and should be interpreted as a layer of thickness $\varepsilon_c L$ within which σ_{zz} is of the order of the total force $\sigma_0 L \sin(z/L)$ divided by the thickness $\varepsilon_c L$.

The estimates of this section imply that a stress concentration layer has an energy density which is $0(1)$. For example, for a singular fiber, the associated energy density is

$$\frac{\sigma_{zz}^2}{2E} = 0 \left(\frac{\sigma_0^2}{2E \varepsilon_t^2} \right) = 0 \left(\frac{\sigma_0^2}{2G} \right) = 0(1), \tag{5.8}$$

since $\varepsilon_t^2 \cong G/E$ according to (3.3). Thus, stress concentration layers contribute negligibly to the total energy because they occupy a negligible amount of the total volume.

6. Stress Penetration

We now turn to the case of a half-space $z \geq 0$ with a sinusoidally varying normal load on the boundary:

$$\sigma_{zz} = \sigma_0 \cos \frac{x}{L}, \quad \sigma_{zx} = 0 \quad (z = 0). \tag{6.1}$$

For this problem the idealized theory gives $u = w = 0$, and thus $\sigma_{zx} = 0$. The stress σ_{xx} in normal lines $z = \text{constant}$ is an arbitrary function $f(z)$, constant along each normal line; we arbitrarily set $\sigma_{xx} = 0$. Then

$$\sigma_{zz} = \sigma_0 \cos \frac{x}{L}, \quad \sigma_{zx} = \sigma_{xx} = 0 \quad (z \geq 0). \tag{6.2}$$

The point to be examined here is that the boundary load propagates along the fibers unaltered, to infinite depth. Thus, the idealized theory does not allow σ_{zz} to vanish as $z \rightarrow \infty$.

For the elastic theory, an easy computation yields

$$\chi = \sigma_0 L^2 (1 - \epsilon_i \epsilon_l)^{-1} \cos \frac{x}{L} \left[-\exp \frac{-\epsilon_l z}{L} + \epsilon_l \epsilon_c \exp \frac{-z}{L \epsilon_c} \right]. \tag{6.3}$$

Then the stress components are

$$\left. \begin{aligned} \sigma_{xx} &= \sigma_0 \frac{\epsilon_l}{\epsilon_c} \cos \frac{x}{L} \exp \frac{-z}{L \epsilon_c} + O(\epsilon_l^2), & \sigma_{xz} &= O(\epsilon_l), \\ \sigma_{zz} &= \sigma_0 \cos \frac{x}{L} \exp \frac{-\epsilon_l z}{L} + O(\epsilon_c \epsilon_l). \end{aligned} \right\} \tag{6.4}$$

We first observe from (6.4c) that σ_{zz} does decay with depth, but the distance required for decay to $1/e$ times the boundary value is L/ϵ_l , a very large multiple of the characteristic length L . For each value of z , the result (6.2a) of the idealized theory is the limit of (6.4c) for $\epsilon_l \rightarrow 0$.

The stress σ_{xx} in normal lines is of an ambiguous order ϵ_l/ϵ_c that conceivably might not be negligible. Indeed, (3.5) shows that ϵ_l/ϵ_c might be nearly as large as $1/2 \nu$, which may be appreciable. However, (6.4a) shows that σ_{xx} is negligible outside a boundary layer whose thickness is $O(L \epsilon_c)$. The resultant of σ_{xx} for the whole layer is $O(\sigma_0 L \epsilon_l)$, which is negligible when $\epsilon_l \rightarrow 0$. Thus, σ_{xx} approaches zero pointwise except at $z = 0$, and in the boundary layer its resultant approaches zero. The idealized theory accordingly predicts simply that $\sigma_{xx} = 0$.

We obtain analogous results for stress penetration along normal lines by interchanging ϵ_l with ϵ_c in (6.4). Here we find that, while the idealized theory predicts that a stress σ_{zz} can propagate without attenuation for an infinite distance along a normal line $x = \text{constant}$, the elastic theory predicts that the actual characteristic attenuation length is L/ϵ_c .

7. Penetration of a Line Load

We again consider a half-space $z \geq 0$ with fibers perpendicular to the boundary. A normal load F per unit length is concentrated along the line $x = z = 0$. According to the idealized theory there is no displacement, and the stress is

$$\sigma_{xx} = \sigma_{xz} = 0, \quad \sigma_{zz} = -F \delta(x). \quad (7.1)$$

(As usual, we have chosen $\sigma_{xx} = 0$ arbitrarily.) Thus, the boundary load penetrates along the line $x = 0$, and there is no stress elsewhere.

For easier comparison with the elastic solution we introduce polar coordinates r, θ , with $\theta = 0$ along $x = 0$, so that

$$z = r \cos \theta, \quad x = r \sin \theta. \quad (7.2)$$

Now

$$\delta(x) = \delta(r \sin \theta) = \frac{1}{r} \delta(\theta) \left(r > 0, |\theta| \leq \frac{\pi}{2} \right). \quad (7.3)$$

Hence, (7.1) yields

$$\sigma_{\theta\theta} = \sigma_{\theta r} = 0, \quad \sigma_{rr} = -\frac{F}{r} \delta(\theta). \quad (7.4)$$

The elastic solution can be found by Fourier synthesis from the results in Section 6, but in any event it is well known [7]. When written in terms of the fundamental parameters ε_c and ε_t , the stress components are $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$ and

$$\sigma_{rr} = -\frac{F}{\pi r} \frac{\varepsilon_t (1 + \varepsilon_c \varepsilon_t) \cos \theta}{(\cos^2 \theta + \varepsilon_c^2 \sin^2 \theta) (\sin^2 \theta + \varepsilon_t^2 \cos^2 \theta)}. \quad (7.5)$$

When ε_t is small, the coefficient of $-F/r$ is

$$\frac{1}{\pi} \frac{\varepsilon_t}{\varepsilon_t^2 + \theta^2} + 0(\varepsilon_t). \quad (7.6)$$

Thus, the delta in (7.4) does have a natural interpretation as the limit of the elastic solution for $\varepsilon_t \rightarrow 0$.

We introduce a characteristic length L by defining the stress concentration region as the region in which $|\sigma_{rr}|$ is greater than some arbitrarily chosen value $F/\pi L$. The boundary of this region is found from (7.5) and (7.6) to be given by

$$\frac{r}{L} = \frac{\varepsilon_t}{\varepsilon_t^2 + \theta^2} + 0(\varepsilon_t). \quad (7.7)$$

On this locus, θ is very small until r is small of order $\varepsilon_t L$, and thus $r \cong z$ and $\theta \cong x/z$. Thus, except very close to the origin, (7.7) can be rearranged as

$$(2 \varepsilon_t z - L)^2 + 4 x^2 = L^2 \left(\frac{r}{L} \gg \varepsilon_t \right). \quad (7.8)$$

The stress concentration region is accordingly elliptical, with a major axis, along the fiber $x = 0$, of length L/ε_l , and a minor axis of length L . As in Section 6, the boundary load penetrates to a depth of order L/ε_l , and as in Section 5, the stress concentration layer has a thickness of the order of ε_l times its length.

The corresponding results for the case in which fibers are parallel to the boundary $z = 0$ rather than perpendicular to it are obtained by interchanging ε_c with ε_l . Again defining the stress concentration layer as the region in which the magnitude of the stress exceeds some value $F/\pi L$, we find that the boundary load penetrates along the normal line $x = 0$ to a depth of order L/ε_c , in a layer of thickness ε_c times this length.

8. Tangential Line Load

We finally consider a half-space $x \geq 0$ with a tangential line load F in the positive z -direction at the origin. The idealized theory gives zero displacement and $\sigma_{xx} = \sigma_{xz} = 0$. We can choose to take $\sigma_{zz} = 0$ at interior points, but in the boundary fiber $x = 0$ there must be a resultant force changing by an amount F at the origin, to balance the applied load. We choose the symmetrical solution,

$$\sigma_{zz} = -\frac{1}{2} F \operatorname{sgn} z \delta(x). \quad (8.1)$$

Properties of the elastic solution are predictable from (8.1). There will be a thin layer of high tensile stress in the fibers next to the boundary. The thickness of this layer at a distance z along the boundary will be $O(\varepsilon_l z)$, and the stress in it will be of the order of $-F/2 \varepsilon_l z$.

As it happens, the exact elastic solution is again given by (7.5), with the boundary now given by $\theta = 0$ and $\theta = \pi$. Since the resultant force in the layer along $\theta = 0$ was found to be $-F$ in Section 7, the resultant in the half $\theta \geq 0$ is $-F/2$, in agreement with (8.1), in which $x \geq 0$.

9. Summary of Estimates

We have considered only infinitesimal elastic deformations, and the problems considered are probably the least complicated that could yield any information about stress concentration and stress penetration. Nevertheless, we believe that order of magnitude estimates based on this information will be valid even in much more complex problems involving large deformations and inelastic response.

The fundamental parameter for fibers is ε_l , which is approximately $(G/E)^{1/2}$, according to (3.3). Here G and E are the moduli for shearing and extension along the fiber direction. Singular fibers should be interpreted as thin layers, of width ε_l times their lengths, in which the tensile stress is of the order of the resultant given by the idealized theory, divided by this width. Disturbances can propagate along fibers without marked attenuation to a distance of order $1/\varepsilon_l$ in comparison to the characteristic length defined by the data of the problem.

The conclusions about normal lines are analogous, but with ε_l replaced by the fundamental parameter for normal lines, ε_c .

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Résumé

Une théorie a déjà été proposée au sujet des matériaux à fibres de renforcement où l'on supposait que ces dernières étaient inextensibles et uniformément distribuées dans un composite considéré comme incompressible. Cependant, quelques unes des prédictions de cette théorie semblent être fondamentalement en désaccord avec la théorie classique de l'élasticité. Il est démontré ici que les résultats inattendus de cette théorie correspondent en fait à des cas limites de la théorie classique de l'élasticité pour des matériaux à isotropie transversale.

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