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# Boundary Layers in Fiber-Reinforced Materials

## 1 Introduction

THE mechanical behavior of composite materials consisting of a family of strong fibers embedded in a weaker matrix can be studied by using an idealized theory proposed by Mulhern, Rogers, and Spencer [1].<sup>1</sup> The fibers are treated as *continuously distributed* and *inextensible*, and the composite as *incompressible*. These idealizations were used in a three-dimensional theory of rigid-plastic materials. Later, Pipkin and Rogers [2] generalized the model to arbitrary elastic, plastic, or viscoelastic response in deformations of any magnitude, but with restriction to plane deformations.

The idealized theory makes problems relatively easy to solve [3-7]. Solutions typically exhibit an unusual feature: infinitesimally thin layers of infinite tensile stress. To gain some assurance that thin layers of high stress can occur in materials with small but nonzero extensibility and compressibility, some exact solutions from classical elasticity theory were examined [8], and it was found that the results of the idealized theory are indeed compatible with the predictions of elasticity theory. The elastic equations involve two small parameters, denoted  $\epsilon$  and  $c$  in the present paper (Section 2). With fibers of small extensibility,  $\epsilon$  is small. Stress-concentration layers along fibers have thicknesses of the order of  $\epsilon$ , and the tensile stress in them is of

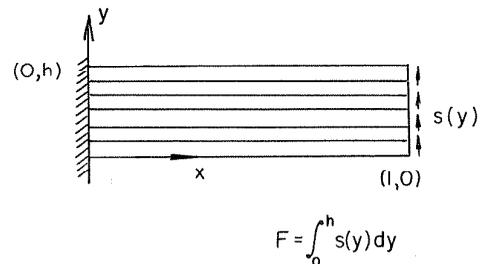


Fig. 1 Cantilever beam with end load

the order of the total load predicted by the idealized theory, divided by this thickness. If the bulk compressibility of the material is also small, then  $c$  is small. In that case, stress-concentration layers orthogonal to fibers can occur. The thickness of such a layer is  $O(c)$ , and the stress in it is of the order of the total load divided by  $c$ .

The idealized theory provides a simple method of determining the gross features of a deformation, and it predicts the existence and locations of stress-concentration layers. The order estimates derived from elasticity theory can then be used to obtain more realistic interpretations of the results. However, there may be cases in which such results are not sufficiently accurate.

In the present paper we develop a method of stress analysis intermediate in accuracy and also intermediate in ease of application between the idealized theory and the exact theory of infinitesimal elastic deformations. Beginning with elasticity theory and treating the extensibility of the material in a preferred direction as a small parameter, we use singular perturbation methods to obtain approximate boundary-layer equations. The problem is reduced to that of solving two Laplace equations and, if higher-order approximations are wanted, a number of Poisson's equations.

The analysis is carried out in the context of a specific problem,

<sup>1</sup> Numbers in brackets designate References at end of paper.

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the plane deformation of a fiber-reinforced cantilever loaded at its outer end, Fig. 1. The analysis of this problem according to the idealized theory, which is trivial, has been described elsewhere [4]. The results are summarized in Section 3. According to the idealized theory, in this problem there are three stress-concentration layers, one on each lateral surface and one on the loaded end. The lateral layers appear because the extensibility  $\epsilon$  is small, and the end layer because of the smallness of the compressibility  $c$ . In the present analysis we make use of approximations based on the smallness of  $\epsilon$ , but leave  $c$  arbitrary until the results are obtained. Thus, although we could analyze the end layer by the same kind of singular perturbation methods that we use for the lateral layers, in the present case the end layer appears as a feature of the so-called interior solution.

Each unknown is represented as the sum of two parts, an *interior* part which represents the solution throughout the bulk of the beam, and a *boundary-layer correction*, which vanishes exponentially with distance from the boundary. The interior solution is obtained (to lowest order) by setting  $\epsilon = 0$  directly in the elastic equations; this yields a Laplace equation for the deflection. The interior solution satisfies the condition of prescribed *normal* traction at the boundary fiber, but *not* the condition of prescribed shearing traction. The boundary-layer correction is required in order to bridge the gap between the interior shearing stress and that prescribed on the boundary.

Within the boundary layer (Section 6), the solution is obtained by first scaling all variables with appropriate powers of  $\epsilon$  and then setting  $\epsilon = 0$ . The displacement parallel to the boundary is found to be harmonic. The normal stress and the displacement normal to the boundary are effectively constant through the thickness of the boundary layer.

The interior solution (Section 8), which includes the solution for the end layer, confirms the idealized theory and its interpretation by order estimates in every respect. The boundary-layer solution (Section 9) confirms the idealized theory for the most part, but brings out an important failing of the idealized theory: At the corners where the boundary layers meet the supporting wall, elasticity theory predicts points of infinite stress. In the idealized theory, these points are buried within extended stress concentrations, and there is no indication that the stress is more severe at such points than at any other part of the stress-concentration layer. Fortunately, the existence of point singularities at corners is to be expected from experience with ordinary isotropic stress analysis. Thus, in interpreting results of the idealized theory, one should expect stress singularities at points where the stress would be singular in an isotropic material; at these points, the estimate of stress as load divided by layer thickness is not valid.

In the present paper, we obtain only the lowest-order approximate solution, but the theory is set up in such a way as to facilitate extension to arbitrarily high orders of approximation. The present paper is based on a dissertation [9] in which such higher-order approximations are discussed in more detail.

## 2 Plane Elastic Deformations of Transversely Isotropic Materials

We consider materials composed of straight, parallel fibers distributed uniformly in a matrix of more compliant material. Macroscopically, such composites can be treated as homogeneous and transversely isotropic about the fiber direction. For infinitesimal elastic response, with fibers parallel to the  $x$ -axis, the stress-strain relations are

$$\begin{aligned} \epsilon_{xx} &= (G/E)\sigma_{xx} - (\nu G/E)\sigma_{yy} - (\nu G/E)\sigma_{zz}, \\ \epsilon_{yy} &= -(\nu G/E)\sigma_{xx} + (G/E')\sigma_{yy} - (\nu' G/E')\sigma_{zz}, \\ \epsilon_{zz} &= -(\nu G/E)\sigma_{xx} - (\nu' G/E')\sigma_{yy} + (G/E')\sigma_{zz}, \\ 2\epsilon_{xy} &= \sigma_{xy}, \quad 2\epsilon_{xz} = \sigma_{xz}, \quad \epsilon_{yz} = (1 + \nu')(G/E')\sigma_{yz}. \end{aligned} \quad (1)$$

Here, the stress components have been made dimensionless by dividing each one by the shear modulus  $G$  ( $= G_{xy}$ ).

We restrict attention to plane deformations in the  $x, y$  plane. For these cases, from (1) we obtain

$$\begin{aligned} \epsilon_{xx} = u_{,x} &= \epsilon^2(\sigma_{xx} - 2\alpha\sigma_{yy}), \\ \epsilon_{yy} = v_{,y} &= c^2\sigma_{yy} - 2\alpha\epsilon^2\sigma_{xx}, \\ 2\epsilon_{xy} = u_{,y} + v_{,x} &= \sigma_{xy}. \end{aligned} \quad (2)$$

Here  $u$  and  $v$  are the displacement components in the  $x$  and  $y$ -directions. The dimensionless parameters in these equations are defined by

$$\begin{aligned} \epsilon^2 &= \frac{G}{E} \left( 1 - \nu^2 \frac{E'}{E} \right), \quad c^2 = [1 - (\nu')^2] \frac{G}{E'}, \\ 2\alpha &= \nu(1 + \nu') / (1 - \nu^2 E'/E). \end{aligned} \quad (3)$$

For reference, the equilibrium equations are

$$\sigma_{xx,x} + \sigma_{xy,y} = 0, \quad \sigma_{xy,x} + \sigma_{yy,y} = 0. \quad (4)$$

We are interested in materials with small compressibility and very strong fibers. The elastic constants accordingly satisfy [8]

$$G/E \ll 1, \quad E'/E \ll 1, \quad 0 < 1 - \nu' \ll 1. \quad (5)$$

Consequently, the parameters  $\epsilon$  and  $c$  are both small in comparison to unity, and they are given approximately by

$$\epsilon^2 \cong G/E \quad \text{and} \quad c^2 \cong (1 - \nu')2G/E'. \quad (6)$$

In a previous paper [8] it was shown that in plane problems, the equation satisfied by Airy's stress function involves only two independent parameters, denoted  $\epsilon_t$  and  $\epsilon_c$ . When  $\epsilon_t$  is small, these two parameters are practically the same as the present parameters  $\epsilon$  and  $c$ :

$$\epsilon/\epsilon_t = c/\epsilon_c = 1 + O(\epsilon^2). \quad (7)$$

The significance of these parameters with respect to stress-concentration layers has been mentioned in Section 1.

## 3 Cantilever: Idealized Theory

We consider the cantilever shown in Fig. 1. The beam is of length  $L$  and height  $H$ . We make distances and displacements dimensionless by letting  $L$  be the unit of length; thus the beam occupies the region  $0 \leq x \leq 1$ ,  $0 \leq y \leq h = H/L$ . The displacements are zero on the fixed end  $x = 0$ . The sides  $y = 0, h$  are traction-free. The end  $x = 1$  is loaded in shear by a (dimensionless) load  $s(y)$  whose resultant is  $F$

$$\sigma_{xx}(1, u) = 0, \quad \sigma_{xy}(1, u) = s(y), \quad F = \int_0^h s(y)dy. \quad (8)$$

The equations of the idealized theory are obtained by setting  $\epsilon = 0$  in (2). The displacement field is then easily found to be a simple shear

$$u = 0, \quad v = Fx/h. \quad (9)$$

The shearing stress is equal to  $F/h$  at all interior points, so it is discontinuous across the bounding fibers  $y = 0$  and  $y = h$ , and also discontinuous across the bounding normal line  $x = 1$  unless the end traction  $s(y)$  is uniform

$$\begin{aligned} \sigma_{xy}(x, y) &= (F/h)[S(y) - S(y - h)] \\ &\quad - [(F/h) - s(y)]S(x - 1). \end{aligned} \quad (10)$$

Here  $S$  is the unit step function, unity for positive argument, and zero for negative argument.

It then follows from the equilibrium equations that the tensile

stress  $\sigma_{xx}$  is zero except in the bounding fibers, where it is singular, and a similar result is obtained for  $\sigma_{yy}$

$$\sigma_{xx}(x, y) = (F/h)(1-x)[\delta(y) - \delta(y-h)], \quad (11a)$$

$$\sigma_{yy}(x, y) = \left[ Fy/h - \int_0^y s(y')dy' \right] \delta(x-1). \quad (11b)$$

Here  $\delta(x)$  is the Dirac delta. A more detailed discussion of these results has been given elsewhere [4].

## 4 Boundary-Layer Theory

We now consider the same problem under the assumption that the parameter  $\epsilon$  is small but not zero. The parameter  $c$  is left arbitrary, except for a mild restriction imposed by the assumption of positive-definiteness of the strain-energy density. We seek a solution valid asymptotically as  $\epsilon \rightarrow 0$ . After a solution has been obtained we will examine the consequences of taking  $c$  to be small as well.

To simplify matters, we restrict attention to cases in which the end load  $s(y)$  is an even function of  $y - \frac{1}{2}h$ . In such cases,  $\sigma_{xy}$  and  $v$  are even functions, and  $\sigma_{xx}$ ,  $\sigma_{yy}$ , and  $u$  are odd. Accordingly, we can restrict attention to the lower half of the beam. The modifications required when the end load is not symmetric are discussed in the report [9] on which this paper is based.

From the results of the idealized theory (Section 3) and the order estimates obtained from previous problems, we expect that the elastic solution will involve a stress-concentration layer along the boundary  $y = 0$ , with thickness of order  $\epsilon$ . For this region, we introduce a stretched coordinate

$$\eta = y/\epsilon. \quad (12)$$

We anticipate that (2) and (4) can be approximated by simpler equations when  $\epsilon$  is small, but that the simplifications which can be made in the interior region will be different from those which can be made in the boundary-layer region. To avoid the problem of matching the boundary-layer solution to the interior solution, we use the following device [10]. Every unknown function,  $f$ , say, is written as the sum of an interior part  $f_i(x, y)$  whose  $y$ -derivatives are  $O(1)$ , and a boundary-layer correction  $\bar{f}(x, \eta)$  whose  $\eta$ -derivatives are  $O(1)$

$$f = f_i(x, y) + \bar{f}(x, \eta). \quad (13)$$

The part  $\bar{f}$  is conceived as being transcendentally small when  $\epsilon$  approaches zero with  $y$  fixed

$$\bar{f}(x, y/\epsilon) = O(\epsilon^n), \quad (y > 0, \text{ all } n). \quad (14)$$

Then, in particular,

$$\bar{f}(x, \infty) = 0. \quad (15)$$

Thus  $f_i$  is the solution at interior points, and  $f_i + \bar{f}$  is the solution in the boundary layer.

## 5 Interior Equations

Let us separate every unknown into two parts as shown in (13) and then use these expressions in (2) and (4). When  $\epsilon \rightarrow 0$ , at a fixed point  $y > 0$  the boundary-layer corrections approach zero faster than any power of  $\epsilon$ , according to (14). Consequently, the so-called interior variables satisfy (2) and (4) by themselves for  $y > 0$ . Similarly, the interior variables must satisfy the boundary conditions at the ends  $x = 0$  and  $x = 1$  by themselves, but *not* the conditions at  $y = 0$  (and  $y = h$ ). Thus the interior component of the solution satisfies all of the conditions of the original problem except the conditions of zero traction on  $y = 0$  (and  $y = h$ ), where the boundary-layer corrections will be important.

Inspection of (2) and (4) shows that if all variables are  $O(1)$  (i.e., bounded as  $\epsilon \rightarrow 0$ ), then  $u$  is  $O(\epsilon^2)$ . Consequently, we can

obtain a better view of the relative importance of various terms by writing

$$u = \epsilon^2 U(x, y) \quad (16)$$

and treating  $U$  as  $O(1)$ . For the interior components of the variables  $\sigma_{xx}$ ,  $\sigma_{yy}$ ,  $\sigma_{xy}$ , and  $v$ , we write  $\sigma_x$ ,  $\sigma_y$ ,  $\sigma$ , and  $V$ , respectively. Then the equations satisfied by the interior variables are

$$U_{,x} = \sigma_x - 2\alpha\sigma_y, \quad (17a)$$

$$V_{,y} = c^2\sigma_y - 2\alpha\epsilon^2\sigma_x, \quad (17b)$$

$$V_{,x} + \epsilon^2 U_{,y} = \sigma, \quad (17c)$$

$$\sigma_{x,x} + \sigma_{y,y} = 0, \quad (17d)$$

$$\sigma_{x,x} + \sigma_{y,y} = 0. \quad (17e)$$

The terms involving  $\epsilon^2$  in (17) are to be treated as known functions. In the lowest order of approximation they are simply omitted. If a solution to higher order is wanted, at each stage these terms are evaluated by using the best previous approximation. We retain these terms in order to make such an iterative procedure possible, although here we intend to obtain only the lowest order approximation. Higher-order approximations, obtained by expansions in series, are explained in more detail elsewhere [9].

From (17b) and (17c) we find that the shearing stress and normal stress are given in terms of the deflection  $V$  by

$$\sigma = V_{,x} + \epsilon^2 U_{,y}, \quad \sigma_y = c^{-2}(V_{,y} + 2\alpha\epsilon^2\sigma_x). \quad (18)$$

The tensile stress is obtained by integrating (17d) and using the boundary condition  $\sigma_x(1, y) = 0$

$$\sigma_x = \int_x^1 \sigma_{,y} dx = V_{,y}(1, y) - V_{,y}(x, y) + \epsilon^2 \int_x^1 U_{,y} dx. \quad (19)$$

$U$  is then obtained by integrating (17a) subject to  $U(0, y) = 0$

$$U = \int_0^x (\sigma_x - 2\alpha\sigma_y) dx. \quad (20)$$

By using (18) in (17e), we obtain

$$c^2 V_{,xx} + V_{,yy} = -\epsilon^2 (c^2 U_{,x} + 2\alpha\sigma_{x,y}). \quad (21)$$

The boundary conditions of no deflection at  $x = 0$  and prescribed shearing stress at  $x = 1$  yield

$$V(0, y) = 0, \quad (22a)$$

$$V_{,x}(1, y) = s(y) - \epsilon^2 U_{,y}(1, y). \quad (22b)$$

In Section 6 we show that the boundary-layer correction to  $\sigma_{yy}$  is  $O(\epsilon)$ . With  $\sigma_{yy} = \sigma_y + \epsilon\bar{\sigma}_y$ , the condition of zero normal stress on  $y = 0$  implies that

$$V_{,y}(x, 0) = -\epsilon c^2 \bar{\sigma}_y(x, 0) - 2\alpha\epsilon^2 \sigma_x(x, 0). \quad (23)$$

The corresponding condition on  $y = h$  is obtained by symmetry

$$V_{,y}(x, h) = -V_{,y}(x, 0). \quad (24)$$

By setting  $\epsilon = 0$  in (18)-(24), we find that the lowest-order approximation is the same as that obtained by setting  $\epsilon = 0$  in (2), except that (20) yields an approximation  $u = \epsilon^2 U$  more refined than  $u = 0$ , which is of no great interest. Thus the deflection  $V$  can be calculated correct to lowest order simply by omitting all terms involving  $\epsilon$  in the original equations. The results concerning the boundary conditions at  $y = 0$  and  $y = h$  are more important. First, only *one* condition is needed at each of these boundaries, because the order of the system has been lowered. Second, one condition is available, in spite of the intervention of the boundary layers. For, the *normal* stress changes so little through the thickness of the boundary layer

that the interior solution must satisfy the condition of prescribed normal traction by itself, in the lowest order of approximation.

## 6 Boundary-Layer Equations

Since the basic equations of the problem are satisfied both by the full solution and by its interior component, they must also be satisfied by the boundary-layer correction terms. By taking into account that these terms depend on  $y$  only through  $\eta = y/\epsilon$ , from (2) and (4), we obtain

$$\bar{u}_{,x} = \epsilon^2(\bar{\sigma}_{xx} - 2\alpha\bar{\sigma}_{yy}), \quad (25a)$$

$$\bar{v}_{,\eta} = c^2\epsilon\bar{\sigma}_{yy} - 2\alpha\epsilon^2\bar{\sigma}_{xx}, \quad (25b)$$

$$\bar{u}_{,\eta} + \epsilon\bar{v}_{,x} = \epsilon\bar{\sigma}_{xy}, \quad (25c)$$

$$\epsilon\bar{\sigma}_{xx,x} + \bar{\sigma}_{xy,\eta} = 0, \quad (25d)$$

$$\epsilon\bar{\sigma}_{xy,x} + \bar{\sigma}_{yy,\eta} = 0. \quad (25e)$$

We expect the shearing stress to change by an amount independent of  $\epsilon$ , i.e.,  $O(1)$ , from one side of the boundary layer to the other. Then (25d) shows that the tensile stress  $\bar{\sigma}_{xx}$  must be  $O(1/\epsilon)$  for equilibrium, as expected from prior estimates. Let us write

$$\bar{\sigma}_{xx} = \epsilon^{-1}\sigma_x(x, \eta), \quad \bar{\sigma}_{xy} = \sigma(x, \eta), \quad (26)$$

and treat  $\bar{\sigma}_x$  and  $\bar{\sigma}$  as  $O(1)$ . From (25e) it then follows that  $\bar{\sigma}_{yy}$  is  $O(\epsilon)$

$$\bar{\sigma}_{yy} = \epsilon\bar{\sigma}_y(x, \eta), \text{ say.} \quad (27)$$

Thus, as mentioned in Section 5, the normal stress changes by a negligible amount through the thickness of the boundary layer, so boundary conditions on  $\sigma_{yy}$  can be applied directly to the interior solution.

By using (26) and (27) in (25a), we find that  $\bar{u}$  is  $O(\epsilon)$

$$\bar{u} = \epsilon\bar{U}(x, \eta). \quad (28)$$

Although this displacement, which involves fiber-stretching, is small, it is nevertheless sufficient to produce the  $O(\epsilon^{-1})$  tensile stress. On the interior, where  $u = O(\epsilon^2)$ , fiber-stretching is much less significant.

By using (26) and (27) in (25b), we find that  $\bar{v}$  is  $O(\epsilon^2)$

$$\bar{v} = \epsilon^2\bar{V}(x, \eta). \quad (29)$$

Thus the boundary-layer correction to the deflection is insignificant. Deflections measured on the surface yield interior values directly, if  $O(\epsilon^2)$  terms are disregarded.

As a check, we verify that the order estimates (26) to (29) are consistent with (25c), which was not used in obtaining them.

By using (26) in (25a) and (25c) we obtain

$$\bar{\sigma}_x - \bar{U}_{,x} + 2\alpha\epsilon^2\bar{\sigma}_y, \quad \bar{\sigma} - \bar{U}_{,\eta} + \epsilon^2\bar{V}_{,x}. \quad (30)$$

On integrating (25e) subject to the boundary condition (15), we obtain

$$\bar{\sigma}_y = \int_{\eta}^{\infty} \bar{\sigma}_{,xx} d\eta = -\bar{U}_{,x} + \epsilon^2 \int_{\eta}^{\infty} \bar{V}_{,xx} d\eta. \quad (31)$$

Thus all stresses in the boundary layer can be calculated (to lowest order) by disregarding  $\bar{V}$  and treating  $u = \epsilon\bar{U}$  as the only displacement. The correction  $\bar{V}$ , if wanted, is found by integrating (25b) subject to (15)

$$\bar{V} = - \int_{\eta}^{\infty} (c^2\bar{\sigma}_y - 2\alpha\bar{\sigma}_x) d\eta. \quad (32)$$

The equation satisfied by  $\bar{U}$  is found by using (30) in (25d)

$$\bar{U}_{,xx} + \bar{U}_{,\eta\eta} = -\epsilon^2(\bar{V}_{,\eta} + 2\alpha\bar{\sigma}_y)_{,x}. \quad (33)$$

The boundary conditions on  $\bar{U}$  include the zero-displacement condition at  $x = 0$ , the condition (15), and the condition of no tensile stress on the end  $x = 1$

$$\bar{U}(0, \eta) = \bar{U}(x, \infty) = 0, \quad \bar{U}_{,x}(1, \eta) = -2\alpha\epsilon^2\bar{\sigma}_y(1, \eta). \quad (34)$$

In addition, the shearing stress  $\sigma_{xy} = \sigma + \bar{\sigma}$  must vanish at  $\eta = 0$ . With (30), this implies that

$$\bar{U}_{,\eta}(x, 0) = -\sigma(x, 0) - \epsilon^2\bar{V}_{,x}(x, 0). \quad (35)$$

To obtain the lowest-order boundary-layer solution, we set  $\epsilon = 0$  in (30) to (35). Unlike the problem of finding the lowest-order interior solution, here the boundary data are not all given at the outset. The interior solution must be found first, in order that its associated boundary shear stress  $\sigma(x, 0)$  may be evaluated. If this were zero, then of course no boundary-layer correction would be needed.

## 7 Discrepancies at Corners

Scaled variables have been introduced in Sections 5 and 6. The original unknowns are expressed in terms of them by

$$u = \epsilon^2 U + \epsilon \bar{U}, \quad v = V + \epsilon^2 \bar{V}, \quad (36)$$

$$\sigma_{xx} = \sigma_x + \epsilon^{-1} \bar{\sigma}_x, \quad \sigma_{yy} = \sigma_y + \epsilon \bar{\sigma}_y, \quad \sigma_{xy} = \sigma + \bar{\sigma}.$$

Two of the boundary conditions of the problem are not satisfied exactly. Consider first the condition  $v(0, y) = 0$ . According to (22a), the interior component satisfies  $V(0, y) = 0$ , but in general  $\bar{V}(0, \eta) \neq 0$ . Thus, near the corners  $x = 0$ ,  $y = 0, h$ , the deflection fails to satisfy the zero-displacement boundary condition. The error, which is  $O(\epsilon^2)$  and confined to very small regions, cannot be removed by carrying the present method of approximation to higher orders. A separate analysis of the corners, which we have not undertaken, would be required.

At the outer corners,  $x = 1, y = 0, h$ , the condition  $\sigma_{xy}(1, y) = s(y)$  is not satisfied exactly. For,  $\sigma(1, y) = s(y)$  according to (22b), but in general  $\bar{\sigma}(1, \eta) \neq 0$ . Thus, in regions of width  $O(\epsilon)$  near the corners, the prescribed shearing traction is not exactly attained. The solution obtained corresponds exactly to a traction distribution whose resultant differs by  $O(\epsilon)$  from that prescribed, so this discrepancy is unimportant.

## 8 Solution: Interior and End Layer

The lowest-order approximation to the deflection  $V$  is found by setting  $\epsilon$  equal to zero in (21)–(24). If the traction distribution  $s(y)$  has the Fourier series representation

$$s(y) = F/h + \sum_1^{\infty} S_n \cos(\lambda_n y), \quad \lambda_n = 2\pi n/h, \quad (37)$$

then the solution is

$$V(x, y) = Fx/h + \sum_1^{\infty} S_n \frac{c}{\lambda_n} \frac{\sinh(\lambda_n x/c)}{\cosh(\lambda_n/c)} \cos(\lambda_n y). \quad (38)$$

Let us now consider the nature of the solution when  $c$  is small. With a negligible error near the end  $x = 0$ ,  $V$  can be written in the more transparent form

$$V(x, y) \cong Fx/h + \sum_1^{\infty} S_n (c/\lambda_n) \exp[\lambda_n(x-1)/c] \cos(\lambda_n y). \quad (39)$$

From this expression we see that  $V$  is given by  $Fx/h$ , the solution according to the idealized theory, except in a layer of thickness  $O(ch)$  at the end  $x = 1$ . In this layer  $V$  differs by  $O(c)$  from the idealized solution. Within this thin layer the applied shearing stress  $s(y)$  is redistributed, and it emerges on the interior as a uniform stress  $F/h$ ; from (18) and (39),

$$\sigma(x, y) \cong F/h + \sum_1^{\infty} S_n \exp [\lambda_n(x-1)/c] \cos(\lambda_n y). \quad (40)$$

This redistribution entails transmission of forces by means of very high ( $0(c^{-1})$ ) tensile or compressive stresses  $\sigma_{yy}$ ; from (18) and (39),

$$\sigma_y(x, y) \cong -c^{-1} \sum_1^{\infty} S_n \exp [\lambda_n(x-1)/c] \sin(\lambda_n y). \quad (41)$$

The resultant tension in the end layer can be found with negligible errors by integrating (41) from  $x = -\infty$  to  $x = 1$ . Then, either with the aid of (37) or directly from the  $y$ -component of the equilibrium equation, we find that this resultant is related to the difference between the applied and interior stress by

$$\frac{d}{dy} \int \sigma_y dx = F/h - s(y). \quad (42)$$

Comparison with (11) shows that the resultant given by the idealized theory is exactly the same. The Dirac deltas of the idealized theory are replaced now by exponentials.

It is easy to verify that  $\sigma_x = 0(c)$  both in the end layer and on the interior (from (19) and (39)). For  $x$  not too close to unity, the interior tensile stress is

$$\sigma_x(x, y) = V_{,y}(1, y). \quad (43)$$

The tensile stress that is created in a given fiber by the disturbance in the end layer is thus transmitted along that fiber with no change in the interior region. This result is consistent with a previous prediction [8] that the attenuation length for stress transmission along fibers is  $0(1/\epsilon)$ .

The negligible displacement  $u = \epsilon^2 U$  can be found by using (20). On the interior, where  $\sigma_y$  is zero and  $\sigma_x$  is given by (43),  $U$  is

$$U(x, y) = xV_{,y}(1, y). \quad (44)$$

Provided that the end load is not applied uniformly, so that  $V_{,y}(1, y)$  is not zero, then (44) shows that there is an extremely small flexural component of deformation. If the classical Saint-Venant approximation were blindly applied to the present problem, it is only this flexural component that would be found.

### 9 Solution: Boundary Layer

To obtain the lowest-order approximate solution in the boundary layer along  $y = 0$ , we set  $\epsilon = 0$  in (33)–(35). We take the interior shearing stress  $\sigma(x, 0)$  in (35) to be  $F/h$ , which is consistent with (40) for small  $c$ . We obtain

$$\bar{U}(x, \eta) = (2F/h) \sum_0^{\infty} \mu_n^{-2} \sin(\mu_n x) \exp(-\mu_n \eta), \quad (45)$$

$$\mu_n = (n + \frac{1}{2})\pi.$$

The tensile stress  $\sigma_{xx} = \epsilon^{-1} \bar{\sigma}_x$  is the only part of the boundary-layer solution that we shall discuss. Other features of the solution are similar to the corresponding features of the end layer solution (Section 8). The tensile stress is found by using (45) in (30)

$$\bar{\sigma}_x(x, \eta) = (2F/h) \sum_0^{\infty} \mu_n^{-1} \cos(\mu_n x) \exp(-\mu_n \eta). \quad (46)$$

The resultant tensile stress in the boundary layer is the integral of this from zero to infinity. By performing the integration and then carrying out the resulting summation, we obtain

$$\int_0^{\infty} \bar{\sigma}_x d\eta = (F/h)(1-x), \quad (47)$$

which agrees exactly with the resultant (11a) given by the idealized theory.

According to the rules that have been proposed for interpreting the results of the idealized theory, the tensile stress in the boundary layer should be of the order of the resultant (47), divided by the thickness  $\epsilon$

$$\sigma_{xx} \cong F(1-x)/\epsilon h. \quad (48)$$

This rule works well in the end layer; see (41). However, if we carry out the summation in (46) and then set  $\eta = 0$ , we obtain

$$\sigma_{xx}(x, 0) = \epsilon^{-1} \bar{\sigma}_x(x, 0) = (2F/\epsilon \pi h) \log [\operatorname{ctn}(\pi x/4)]. \quad (49)$$

For  $x$  near unity the boundary stress (49) is the same as the estimate (48), but at the corner  $x = y = 0$ , there is a weak stress singularity not predicted by the estimate. The principal advantage of the methods discussed here, in comparison to the idealized theory, is that the boundary-layer methods are capable of revealing point singularities of this kind.

### 10 Limitations of Theory

It has been pointed out [4] that the solution of the cantilever problem by the classical Saint-Venant approximation makes use of the prior assumption that the deformation is flexural. It was argued on the basis of the idealized theory that this assumption is false for beams reinforced by strong fibers. The present analysis supports the conclusion from the idealized theory that the beam deforms in shear rather than flexure.

However, the present theory has its own limitations. If  $ch$  is as large as unity, the so-called end layer occupies the whole beam, and if  $\epsilon$  is as large as  $h/2$ , the two lateral layers fill the entire body. Thus, for  $c > 1/h$  or  $\epsilon > h/2$ , we expect flexure according to the Saint-Venant theory to be more realistic than the present results. This puts a limitation  $1/c > h > 2\epsilon$  on the values of the slenderness  $h = H/L$  for which the present results will be useful.

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