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STRUCTURAL ANALOGIES FOR SCALAR FIELD PROBLEMS

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SUMMARY

This paper describes how general-purpose finite element structural analysis computer programs can be used, without modification, to solve various scalar field equations such as the wave equation, the Helmholtz equation, Laplace's equation, Poisson's equation, the heat equation and the telegraph equation, as well as mixed field problems (such as coupled structural–acoustic problems) which involve these equations.

INTRODUCTION

This paper develops the analogy between the equations of elasticity and the common equations of classical mathematical physics so that standard general-purpose structural analysis computer codes can be used, without modification, to solve a variety of field problems. This paper shows how to specify the elastic constants and applied loads to solve problems involving equations such as the wave equation, the Helmholtz equation, Laplace's equation, Poisson's equation, the heat equation and the telegraph equation.

The use of finite elements to solve such equations is, of course, not new. For example, Zienkiewicz and Cheung[1] were solving field problems with finite elements some 15 years ago. Also, Martin[2] in 1968 addressed potential flow problems. However, what has not previously appeared, to the author's knowledge, is the recipe for using standard structural codes for solving such problems, in both two and three dimensions.

These analogies have been found useful in a variety of fluid–structure interaction problems, in which the coupling of a structure with an acoustic fluid is the main motivation for solving wave, Helmholtz and Laplace equations with a structural code.[3–5]

Although a knowledge of elasticity is needed to derive the analogies, there is clearly no requirement that a user of the analogies be conversant with structural analysis.

THE ANALOGY

Many linear problems in mathematical physics involve the solution of an equation obtained by specializing the general form

$$\nabla^2\phi + g = a\ddot{\phi} + b\dot{\phi} \tag{1}$$

where ∇^2 is the Laplacian operator, dots denote partial time differentiation, the functions g, a and b are, in general, position-dependent, and the unknown scalar function ϕ depends on both position and time.

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On the other hand, the x-component of the Navier equations of elasticity, which are the equations solved by structural analysis computer programs, is

$$\left(\frac{\lambda+2\mu}{\mu}\right)u_{,xx}+u_{,yy}+u_{,zz}+\left(\frac{\lambda+\mu}{\mu}\right)(v_{,xy}+w_{,xz})+\frac{1}{\mu}f_x=\frac{\rho}{\mu}\ddot{u} \tag{2}$$

where u, v and w are the Cartesian components of displacement, λ and μ are the Lamé elastic constants (where μ is the shear modulus), f_x is the x-component of body force per unit volume, ρ is the mass density, and commas denote partial differentiation.[6]

When equation (2) is compared to equation (1), it is clear that equation (2) reduces to equation (1) if ϕ is represented by u (the x-component of displacement) and

$$\frac{\lambda+2\mu}{\mu}=1 \tag{3}$$

$$v\equiv w\equiv 0 \tag{4}$$

$$\frac{\rho}{\mu}=a \tag{5}$$

$$\frac{1}{\mu}f_x=g-b\dot{\phi} \tag{6}$$

Thus, if the shear modulus μ of a particular finite element is μ_e (which can, as will be seen, be selected arbitrarily), the other material constants are

$$\lambda_e=-\mu_e \tag{7}$$

$$\rho_e=\mu_e a \tag{8}$$

and the body force per unit volume is

$$f_x=\mu_e(g-b\dot{\phi}) \tag{9}$$

The two terms in equation (9) represent, respectively, a gravitational type of force and a distributed dashpot. If the element volumes are simply 'lumped' at the various grid points, the total force F_x acting on a typical grid point to which the volume V is assigned is

$$F_x=\mu_e gV-(\mu_e bV)\dot{\phi} \tag{10}$$

where the dashpot constant is $\mu_e bV$.

If the forcing function g in equations (1) and (10) is independent of position, as, for example, in the classical St. Venant torsion problem, it may be specified conveniently by applying to the structure a gravitational field whose gravitational constant g_0 satisfies

$$\rho_e g_0=\mu_e g \tag{11}$$

In three-dimensional elasticity, the relationships between the Lamé constants and the more conventional engineering constants are

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu=\frac{E}{2(1+\nu)} \tag{12}$$

$$E=\frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \qquad \nu=\frac{\lambda}{2(\lambda+\mu)} \tag{13}$$

where E and ν are Young's modulus and Poisson's ratio, respectively.[6] Equation (13) shows that equation (7) is satisfied only for infinite values of E and ν. Instead, one merely chooses

$$E_e = \alpha\mu_e \qquad \alpha \gg 1 \tag{14}$$

$$\nu_o = \frac{E_e}{2\mu_e} - 1 \cong \frac{\alpha}{2} \tag{15}$$

Large values for these constants do not create any numerical problems since most finite element programs use E_e and ν_e only to compute the coefficients which appear in the stress–strain relations. Equations (12)–(15) indicate that α should be chosen large enough so that $\alpha+1$ is indistinguishable (numerically) from α. Thus, $\alpha = 10^{20}$ suffices on most computers.

To summarize, equation (1) can be solved with elasticity finite elements if the three-dimensional region is modelled with 3-D ('solid') finite elements having

$$E_e = \alpha\mu_e \qquad \nu_e = \frac{\alpha}{2} \qquad \rho_e = \mu_e a \quad (\alpha \gg 1) \tag{16}$$

Although the preceding derivation chose u, the x-component of displacement, to represent the scalar field variable ϕ (with the other two components v and w constrained to zero), any of the three Cartesian displacement components could be used. The shear modulus μ_e in equation (16) is arbitrary since all dimensional material constants, as well as the load given by equation (10), are proportional to it. Consequently, it is convenient in many cases to choose $\mu_e = 1$.

Axisymmetric problems

The solution of three-dimensional problems in cylindrical co-ordinates follows the same approach as in Cartesian co-ordinates, except that the z-component of displacement is the only component which may be used to represent the scalar field variable ϕ. It can easily be shown that neither the r- nor the θ-component of the Navier equations can be specialized to yield equation (1).

Two-dimensional problems

The development of the analogy in two-dimensional (plane stress) elasticity is similar to that in three dimensions but results in different values for the material constants E and ν. It is convenient to start from equation (7), from which it follows that the 6×6 material matrix $\mathbf{G}$ relating the six stress components to the corresponding strain components is the coefficient matrix in

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \mu_e \begin{bmatrix} 1 & -1 & -1 & & & \\ -1 & 1 & -1 & & & \\ -1 & -1 & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{Bmatrix} u_{,x} \\ v_{,y} \\ w_{,z} \\ u_{,y}+v_{,x} \\ v_{,z}+w_{,y} \\ w_{,x}+u_{,z} \end{Bmatrix} \tag{17}$$

The two-dimensional equivalent of this 3-D material matrix is

$$\mathbf{G} = \mu_e \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{18}$$

If equation (18) is compared to the material matrix for plane stress,[6]

$$\mathbf{G}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix} \tag{19}$$

the material constants for the finite element are

$$E_e=\beta\mu_e \qquad \nu_e=\tfrac{1}{2}\beta-1 \qquad \rho_e=\mu_e a \quad (\beta \ll 1) \tag{20}$$

where μ_e is the shear modulus, and β must be small compared to unity but not so small that $1+\beta$ is indistinguishable (numerically) from unity (e.g. $\beta=10^{-5}$).

To summarize, the two-dimensional form of equation (1) can be solved with elasticity finite elements if the 2-D region is modelled with 2-D plane stress (membrane) finite elements whose material constants satisfy equation (20). In Cartesian co-ordinates, either the x- or y-component of displacement may be chosen to represent the scalar field variable ϕ, with the other component constrained to zero everywhere. The shear modulus μ_e in equation (20) is arbitrary, as it is in 3-D, and hence may be conveniently taken as unity.

BOUNDARY CONDITIONS

Most boundary conditions likely to be encountered in connection with equation (1) will probably be special cases of the general form

$$a_1\frac{\partial\phi}{\partial n}+a_2\phi+a_3\dot{\phi}+a_4\ddot{\phi}+a_5=0 \tag{21}$$

where dots denote partial time differentiation, and n is the outward normal. To relate this boundary condition to the elasticity analogy, we first examine the normal derivative.

The normal derivative

Let $\mathbf{n}$ denote the unit outward normal from the region at a surface point. The goal is to evaluate the normal derivative $\partial\phi/\partial n$. When ϕ is replaced with its structural analogue u (the x-component of displacement),

$$\frac{\partial u}{\partial n}=\nabla u\cdot\mathbf{n}=u_{,x}n_x+u_{,y}n_y+u_{,z}n_z \tag{22}$$

Using the constitutive equation (17) and the constraint (4) yields

$$\frac{\partial u}{\partial n}=(\sigma_{xx}n_x+\sigma_{xy}n_y+\sigma_{xz}n_z)/\mu_e \tag{23}$$

where the parenthetical expression is equal to the x-component of the stress vector $\mathbf{T}^{(n)}$ acting on a surface with unit outward normal $\mathbf{n}$.[6] Hence,

$$\frac{\partial u}{\partial n}=\frac{T_x^{(n)}}{\mu_e} \tag{24}$$

If the surface is discretized with finite elements, the surface traction $T_x^{(n)}$ can be replaced by its lumped equivalent F_x/A, where F_x is the x-component of the force applied to a particular point (on the surface with outward unit normal $\mathbf{n}$) to which the area A has been assigned. Hence, we obtain the final expression for the normal derivative as

$$\frac{\partial u}{\partial n}=\frac{F_x}{\mu_e A} \tag{25}$$

In other words, to enforce the outward normal derivative $\phi_{,n}$ at a surface point, one applies a 'load' at that point equal to $\mu_e A\phi_{,n}$. A positive force F_x corresponds to a positive outward normal derivative.

Specification of boundary conditions

From equation (25), the boundary condition (21) is enforced by applying a 'load' to each boundary point (to which the area A has been assigned) given by

$$F_x=-\frac{\mu_e A}{a_1}(a_2\phi+a_3\dot{\phi}+a_4\ddot{\phi}+a_5) \qquad a_1\neq 0 \tag{26}$$

The a_2 term is analogous to a scalar spring of constant $\mu_e A a_2/a_1$ connected between the point and ground. The a_3 term is analogous to a scalar dashpot of constant $\mu_e A a_3/a_1$ connected between the point and ground. The a_4 term is analogous to an added mass† of value $\mu_e A a_4/a_1$. The a_5 term is a time-independent force given by $-\mu_e A a_5/a_1$.

As expected, the special case of the Neumann boundary condition ($\phi_{,n}=0$) corresponds to the traction-free boundary in elasticity and hence is a natural boundary condition. The Dirichlet condition ($\phi=\phi_0$) is implemented merely by enforcing the desired value as a 'displacement' boundary condition.

FLUID–STRUCTURE INTERACTION

The analogue presented can also be applied productively to solve various problems in which structures and fluids interact,[3–5] including, for example, the determination of the natural frequencies of vibration of submerged structures, of tanks containing liquids, or the calculation of the response of such fluid–structure systems to sinusoidal and general transient excitations.

For these problems, the fluid is generally treated as an acoustic medium,[8–10] a fluid whose pressure p satisfies the wave equation

$$\nabla^2 p=\ddot{p}/c^2 \tag{27}$$

where c is the speed of sound in the fluid. The boundary condition at a fluid–structure interface can be obtained from momentum and continuity considerations:

$$\frac{\partial p}{\partial n}=-\rho\ddot{u}_n \tag{28}$$

where n is the normal at the interface and ρ is the mass density of the fluid.

Sufficient tools have been developed in the preceding sections to solve with finite elements problems in which structures interact with acoustic fluids. The structure can be modelled with finite elements in the usual way to yield the matrix equation

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}\mathbf{u}=\mathbf{F}(t) \tag{29}$$

† Here, one should probably use a consistent, rather than lumped, formulation since Zarda and Marcus[7] showed that the differences between the two are not insignificant for free surface flow problems.

On the basis of the , nalogue developed above, the fluid region, if finite, can also be modelled with isotropic 3-D ('solid') finite elements for which

$$\mu_e = 1 \qquad E_e = 10^{20} \qquad \rho_e = 1/c^2 \tag{30}$$

according to equation (16). The resulting finite element equation for the acoustic fluid region is thus of the form

$$\mathbf{Q}\ddot{\mathbf{p}} + \mathbf{H}\mathbf{p} = \mathbf{0} \tag{31}$$

where **p** is the vector of fluid pressures at the grid points, and **H** and **Q** are the 'stiffness' and 'mass' matrices for the fluid, respectively.

At the fluid–structure interface, the condition (28) requires the application of a 'force' to each interface pressure node given by $\rho A\ddot{u}_n$, where A is the area lumped at that node, and $\ddot{u}_n$ is the normal component of fluid (and structural) particle acceleration at the node (**n** is defined here as the unit outward normal from the structure rather than from the fluid). The reverse coupling, that of the fluid pressure p on the structure, is merely that of a normal force equal to $-pA$ applied to each interface structural node.

Since these interface loads are proportional to fundamental unknowns or their derivatives, they can be moved to the left-hand sides of equations (29) and (31) to yield a system of the general form (with $\mu_e = 1$)

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ -\rho\mathbf{A}^{\mathrm{T}} & \mathbf{Q} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{A} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ \mathbf{0} \end{Bmatrix} \tag{32}$$

where **A** is an area matrix with non-zeros only at interface degrees-of-freedom. The system (32) is the 'lumped' equivalent of a similar system obtained by Zienkiewicz and Newton[8] with a consistent formulation.

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