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LETTER TO THE EDITOR

A SYMMETRIC POTENTIAL FORMULATION FOR FLUID-STRUCTURE INTERACTION

1. INTRODUCTION

Dynamics problems involving the interaction between an elastic structure and an acoustic fluid (for which the pressure satisfies the scalar wave equation) have been formulated for finite element solution [1] by using either pressure [2–5] or fluid particle displacement [6–9] as the fundamental unknown in the fluid region. In three dimensions, the pressure and displacement formulations result in, respectively, one and three degrees of freedom per finite element mesh point. Thus the pressure approach has the advantage of fewer unknowns and a smaller overall matrix profile or bandwidth. On the other hand, the displacement approach results in symmetric coefficient matrices (in contrast to the pressure formulation, for which the matrices are non-symmetric) and a fluid–structure interface condition which is easier to implement on general purpose finite element computer programs. However, the displacement approach also suffers from the presence of spurious resonances [9], a situation which can be bothersome in time-harmonic problems, either forced or unforced. The purpose of this note is to point out that the principal disadvantage of the pressure formulation, non-symmetric coefficient matrices, can be removed merely by reformulating the pressure solution approach so that a velocity potential rather than pressure is used as the fundamental unknown in the fluid region. For some situations, particularly steady state problems involving damped systems and time-dependent problems, significant computational advantages result.

2. THE PRESSURE FORMULATION

One well-known finite element approach [1–5] for solving coupled fluid–structure interaction problems in which pressure is used as the nodal unknown in the fluid region results in a matrix equation of the form

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ -\rho\mathbf{A}^{\mathrm{T}} & \mathbf{Q} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{p}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{A} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{Bmatrix}, \tag{1}$$

where $\mathbf{u}$ is the vector of displacement components in the structure, $\mathbf{p}$ is the vector of fluid pressures at the nodes of the fluid region, $\mathbf{M}$ is the structural mass matrix, $\mathbf{K}$ is the structural stiffness matrix, $\mathbf{Q}$ is the fluid "inertia" matrix, $\mathbf{H}$ is the fluid "stiffness" matrix, $\mathbf{A}$ is the area matrix which converts fluid pressures at interface points to structural loads, $\mathbf{B}$ is the symmetric damping matrix for the structure, $\mathbf{C}$ is the symmetric damping matrix for the fluid (which arises, for example, from a radiation condition) and ρ is the fluid mass density. Either or both right-hand side forcing functions $\mathbf{f}_1$ and $\mathbf{f}_2$ may be present in some situations. For example, in the eigenvalue problem both $\mathbf{f}_1$ and $\mathbf{f}_2$ vanish. If the excitation is applied directly to the structure, $\mathbf{f}_1$ is non-zero and $\mathbf{f}_2$ vanishes. If the excitation is applied directly to a contained fluid, $\mathbf{f}_1$ vanishes and $\mathbf{f}_2$ is non-zero. For steady-state and transient scattering problems involving exterior fluids, the variable $\mathbf{p}$ generally represents only the scattered component of the total pressure, and $\mathbf{f}_1$ and $\mathbf{f}_2$ are proportional to the free field incident pressure and incident fluid particle acceleration, respectively, at the interface.

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Equation (1) can be solved with some off-the-shelf general purpose structural codes (e.g., NASTRAN) since the matrices **Q** and **H** can be assembled by modeling the fluid region with elastic finite elements whose material properties and constraints are assigned in such a way that the equations of elasticity reduce to the scalar wave equation [10]. The coupling terms in the **A** matrix must be entered directly [4].

3. A SYMMETRIC POTENTIAL FORMULATION

The non-symmetry of the coefficient matrices in equation (1) can be removed by reformulating the problem in terms of a new unknown vector **q** such that

$$\mathbf{p} = \dot{\mathbf{q}}. \tag{2}$$

If the second partition of equation (1) is divided by $-\rho$ and integrated in time, and **p** is replaced by $\dot{\mathbf{q}}$, one obtains the symmetric form

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & -\mathbf{Q}/\rho \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{q}} \end{Bmatrix} + \begin{bmatrix} \mathbf{B} & \mathbf{A} \\ \mathbf{A}^{\mathrm{T}} & -\mathbf{C}/\rho \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{q} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{H}/\rho \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{q} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_1 \\ -\mathbf{g}_2/\rho \end{Bmatrix}, \tag{3}$$

where

$$\mathbf{g}_2(t) = \int_0^t \mathbf{f}_2(\tau)\, \mathrm{d}\tau. \tag{4}$$

The new variable **q** is, except for a multiplicative constant, the velocity potential φ long used by fluid dynamicists, since

$$\mathbf{p} = -\rho\dot{\varphi}. \tag{5}$$

Equation (3) could also be recast in terms of φ rather than **q** as the fundamental fluid unknown, but no particular advantage would result. In fact, the use of **q** rather than φ has the slight practical advantage that the fluid pressure can be recovered directly from the finite element program as the time derivative (velocity) of the unknown **q**.

4. REMARKS

The only effect on the equations of switching to a velocity potential as the unknown has been to move the coupling matrix **A** from the mass and stiffness parts of the problem to the damping part of the problem. If a problem already has damping present from another source (e.g., structural damping or radiation damping), the reformulation has yielded symmetric matrices with no penalty.

However, for problems for which both damping matrices (**B** and **C**) are zero, the reformulation (3) may offer no advantages over equation (1). For example, for eigenvalue problems ($\mathbf{f}_1 = \mathbf{f}_2 = \mathbf{0}$), the reformulation converts an undamped, non-symmetric problem to one which is "damped" and symmetric. Both of these problems require a complex eigenvalue extraction routine to solve, even though the eigenvalues turn out to be real. It is thus not surprising that for low frequency eigenvalue calculations the fluid is frequently assumed to be incompressible ($\mathbf{Q} = \mathbf{0}$) so that the pressure vector **p** in the undamped eigenvalue problem arising from equation (1) can be eliminated from the system to yield a simpler eigenvalue problem [1, 2, 5]

$$(\mathbf{M} + \rho\mathbf{A}\mathbf{H}^{-1}\mathbf{A}^{\mathrm{T}})\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}. \tag{6}$$

In equation (6), the matrix

$$\mathbf{M}_a = \rho\mathbf{A}\mathbf{H}^{-1}\mathbf{A}^{\mathrm{T}} \tag{7}$$

is referred to as the added mass matrix.

For forced time-harmonic problems which are undamped ($\mathbf{B} = \mathbf{C} = \mathbf{0}$), the reformulation results in a system *more* costly to solve, because the time-harmonic form of equation (1) requires a non-symmetric, real solution algorithm, in contrast to the symmetric, complex algorithm required by the time-harmonic form of equation (3) (with $\mathbf{B} = \mathbf{C} = \mathbf{0}$). Non-symmetric matrix operations cost about twice as much to perform on a computer as symmetric operations, whereas complex arithmetic costs about four times as much as real arithmetic.

For the direct integration of undamped ($\mathbf{B} = \mathbf{C} = \mathbf{0}$) time-dependent problems, the reformulation results in a net gain for two reasons. First, a non-symmetric, real problem has been converted to a symmetric, real problem. The only price paid for this conversion is that the coefficient matrices involved in the integration algorithm (such as the Newmark-beta variant used in NASTRAN) now involve linear combinations of mass, stiffness, and damping rather than only mass and stiffness. The second reason for preferring the reformulation for transient problems is that for scattering problems the "loading" vector $\mathbf{f}_2$ in equations (1) and (3) is proportional to the incident fluid particle acceleration at the fluid–structure interface. But incident loadings with a pressure discontinuity (a common occurrence) also have a particle velocity discontinuity and thus a particle acceleration which is a Dirac delta function. Since at every interface point the Dirac delta function would occur at a different time, the impulsive right-hand side cannot simply be replaced by non-zero initial conditions on $\dot{\mathbf{p}}$. The difficulty is eliminated with the reformulation, however, because the right-hand side of equation (3) involves the *integral* of $\mathbf{f}_2$, which is proportional to the incident fluid particle velocity, which in this case has a simple discontinuity.

Thus, for transient problems and for steady state problems involving damped systems, the velocity potential formulation has significant computational advantages over the pressure formulation.

David W. Taylor Naval Ship Research and Development Center, Bethesda, Maryland 20084, *U.S.A.* G. C. EVERSTINE

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