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FINITE ELEMENT SOLUTION OF TORSION AND OTHER 2-D POISSON EQUATIONS

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SUMMARY

The NASTRAN structural analysis computer program may be used, without modification, to solve two-dimensional Poisson equations such as arise in the classical St. Venant torsion problem. The nonhomogeneous term (the right-hand side) in the Poisson equation can be handled conveniently by specifying a gravitational load in a "structural" analysis. The use of an analogy between the equations of elasticity and those of classical mathematical physics is summarized in detail.

BACKGROUND

A computer program like NASTRAN has such wide-ranging capability and versatility that it can be applied in areas other than those for which it was specifically designed by drawing analogies between the equations which NASTRAN solves (e.g., the equations of elasticity) and those of other application areas. For example, before NASTRAN had an explicit heat transfer capability, Mason (Ref. 1) described ways to solve two-dimensional heat conduction problems using the standard structural capabilities. Later, it was shown (Ref. 2-4) that NASTRAN could be directly applied to certain fluid-structure interaction problems such as underwater vibrations and shock. More recently, it has been shown (Ref. 5) that general purpose finite element computer programs can be used, as is, to solve various scalar field equations such as the wave equation, the Helmholtz equation, Laplace's equation, Poisson's equation, the heat equation, and the telegraph equation, as well as mixed field problems (such as coupled structural-acoustic problems) which involve these equations. It was, in particular, shown how to specify the elastic constants, boundary conditions, and applied loads to solve problems involving these equations.

The use of the analogy developed in Ref. 5 will be summarized in detail and illustrated for the classical problem of torsion of prismatic bars, which requires the solution of the two-dimensional Poisson equation.

THE ANALOGY

Many linear problems in mathematical physics involve the solution of an equation obtained by specializing the general form

$$\nabla^2\phi + g = a\ddot{\phi} + b\dot{\phi} \tag{1}$$

where ∇^2 is the Laplacian operator, dots denote partial time differentiation, the functions g, a, and b are, in general, position-dependent, and the unknown scalar function ϕ depends on both position and time.

Special cases of Equation (1) arise in such diverse applications as heat conduction, acoustics, electrical and magnetic potential problems, torsion of prismatic bars, potential fluid flow, and seepage through porous media. Several common special cases are listed here:

Laplace's equation:	$\nabla^2\phi = 0$	(2)
Poisson's equation:	$\nabla^2\phi + g = 0$	(3)
wave equation:	$\nabla^2\phi = \ddot{\phi}/c^2$	(4)
heat equation:	$k\nabla^2\phi + q = \rho c \dot{\phi}$	(5)
telegraph equation:	$\partial^2\phi/\partial x^2 = LC\ddot{\phi} + RC\dot{\phi}$	(6)
Helmholtz equation:	$\nabla^2\phi + k^2\phi = 0$	(7)

The Helmholtz equation is the time-harmonic form of the wave equation.

Most boundary conditions likely to be encountered in connection with Equation (1) will probably be special cases of the general form

$$a_1\,\partial\phi/\partial n + a_2\phi + a_3\dot{\phi} + a_4\ddot{\phi} + a_5 = 0 \tag{8}$$

where n is the outward normal at the boundary. For example, in heat conduction problems, a boundary with a prescribed temperature function ϕ_0 satisfies the Dirichlet condition

$$\phi = \phi_0 \tag{9}$$

and a perfectly insulated boundary has the Neumann condition

$$\partial\phi/\partial n = 0 \tag{10}$$

In free surface flow problems, the linearized free surface condition on the velocity potential is (Ref. 6)

$$\ddot{\phi} + g_0 \phi_{,z} = 0 \tag{11}$$

where g_0 is the acceleration due to gravity, the free surface is the plane z=constant, and commas denote partial differentiation. In radiation problems, the one-dimensional (plane wave) radiation condition that the velocity potential must satisfy at a non-reflecting boundary is

$$\phi_{,n} + \dot{\phi}/c = 0 \tag{12}$$

where c is the wave speed.

An example of a boundary condition not of the general form of Equation (8) is the condition which must be satisfied at an accelerating boundary of a fluid

$$p_{,n} + \rho \ddot{u}_n = 0 \tag{13}$$

where p is the fluid pressure, ρ is the mass density, and $\ddot{u}_n$ is the outward normal component of fluid particle acceleration.

According to the analogy (Ref. 5) between Equation (1) and the Navier equations of classical elasticity, Equations (1) and (8) can be solved with elastic finite elements using the following procedure:

1. Select one of the three Cartesian components of displacement (or the z-component in cylindrical coordinates) to represent the scalar field variable ϕ. Constrain all other displacement components everywhere in the field.

2. Model the domain of interest (either 2-D or 3-D) with finite elements having material constants satisfying

$$E_e = \alpha G_e , \qquad \rho_e = a G_e \tag{14}$$

where "a" is the variable appearing in Equation (1), and E_e, G_e, and ρ_e denote the Young's modulus, shear modulus, and mass density assigned to the finite elements on the material card (MAT1 in NASTRAN). The subscript "e" has been added to emphasize that these constants are merely numbers assigned to the elements and may bear no resemblance to any actual material porperties associated with a particular application. The dimensionless constant α in Equation (14) should, for 3-D problems, be chosen large enough to make α+1 numerically indistinguishable from α (Ref. 5). For 2-D problems, α should be small, but not so small that $1+\alpha$ is numerically indistinguishable from unity. Thus, on most computers (Ref. 5),

$$\alpha = \begin{cases} 10^{-5} & \text{(2-D)} \\ 10^{20} & \text{(3-D)} \end{cases} \tag{15}$$

The shear modulus G_e can be selected arbitrarily (Ref. 5). The finite elements eligible for use in the model are those derived from classical elasticity theory rather from the engineering theories involving beams, plates, or shells. Thus, for 2-D problems, the plane stress membrane elements (such as QDMEM1 or TRIM6 in NASTRAN) are appropriate. For 3-D problems, the solid elements (such as IHEXi or TRAPRG in NASTRAN) should be used.

3. Apply to the unconstrained degree of freedom (DOF) at each grid point in the region a "force" given by

$$F = G_e \; g \; V \tag{16}$$

where V is the volume assigned to the point and g is the function appearing in Equation (1). For problems for which the function g in Equation (1) is independent of position (as, for example, in the classical St. Venant torsion problem), this load may be specified conveniently by applying to the "structure" a gravitational field for which the acceleration due to gravity g_0 satisfies

$$\rho_e \, g_0 = G_e \, g \tag{17}$$

4. Connect between ground and the unconstrained DOF at each grid point in the region a scalar dashpot (e.g., DAMP1 in NASTRAN) whose damping constant (the ratio of damping force to velocity) is $G_e bV$, where b is the function appearing in Equation (1) and V is the volume assigned to the point.

5. Enforce the boundary condition (8) by applying to the unconstrained DOF at each grid point on the boundary of the region a "force" given by

$$F = -G_e \, A \, (a_2 \, \phi + a_3 \, \dot{\phi} + a_4 \, \ddot{\phi} + a_5)/a_1 \, , \qquad a_1 \neq 0 \tag{18}$$

where A is the area assigned to the point. (In general (Ref. 5), the outward normal derivative $\partial\phi/\partial n$ is enforced at a boundary point by applying a "force" to the unconstrained DOF at that point equal to $G_e \, A \, \partial\phi/\partial n$. A positive force corresponds to a positive outward normal derivative.) In

Equation (18), the a_2 term is analogous to a scalar spring of constant G_eAa_2/a_1 connected between the point and ground. The a_3 term is analogous to a scalar dashpot of constant G_eAa_3/a_1 connected between the point and ground. The a_4 term is analogous to an added mass of value G_eAa_4/a_1 attached to the point. (Here, one should probably use a consistent, rather than lumped, formulation since Zarda and Marcus (Ref. 6) showed that the differences between the two are not insignificant for free surface flow problems.) The a_5 term is a time-independent force given by $-G_eAa_5/a_1$. As expected, the special case of the Neumann boundary condition ($\phi_{,n}=0$) corresponds to the traction-free boundary in elasticity and hence is a natural boundary condition. The Dirichlet condition ($\phi=\phi_0$) is implemented merely by enforcing the desired value as a "displacement" boundary condition.

EXAMPLE: TORSION OF PRISMATIC BARS

A simple example involving the torsion of prismatic bars can illustrate the use of the structural analogy. The stress distribution over a non-circular cross section of a twisted bar is determined by finding the stress function $\phi(x,y)$ which satisfies the two-dimensional Poisson equation

$$\nabla^2\phi = -2G\theta \tag{19}$$

in the cross section and is zero on the boundary. The stresses of interest are obtained by differentiation:

$$\tau_{xz} = \partial\phi/\partial y , \qquad \tau_{yz} = -\partial\phi/\partial x \tag{20}$$

In Equation (19), G is the shear modulus of the bar, and θ is the angle of twist per unit length. The torsional constant J for the cross section is given by (Ref. 7)

$$J = (2/G\theta) \int_A \phi \, dA \tag{21}$$

The specific cross section considered here is the equilateral triangle of altitude "a" (Figure 1). (This is not the same "a" that appears in Equation (1).) For this region, Equation (19) can be solved in closed form (Ref. 7) to yield

$$\phi = -G\theta\, [(x^2+y^2)/2 - (x^3-3xy^2)/2a - 2a^2/27] \tag{22}$$

Along the x-axis, the stresses are obtained from Equation (20) as

$$\tau_{xz} = 0 \qquad (23)$$

$$\tau_{yz} = 3G\theta\ (2ax/3 - x^2)/2a \qquad (24)$$

The maximum stress occurs at the middle of the sides of the triangle (x=-a/3); hence

$$\tau_{max} = G\ \theta\ a/2 \qquad (25)$$

The torsional constant is obtained by substituting Equation (22) into Equation (21):

$$J = a^4/15\sqrt{3} \qquad (26)$$

For the numerical experiment, the following parameters were chosen:

$$a = 0.09\ \text{m}$$
$$G - 80\ \text{GPa}$$
$$\theta = 0.04\ \text{rad/m}$$

Although symmetry would require modeling only one-sixth of the triangle, the upper half was modeled with the finite element mesh shown in Figure 2. The element used (NASTRAN's IS2D8) is the standard two-dimensional, eight-node, quadratic, isoparametric, plane stress membrane element available in many finite element structural analysis computer programs. In the NASTRAN implementation (Ref. 8) used for this example, nodal stresses are computed by extrapolating from stresses computed directly at the Gauss integration points. When two or more elements are connected to a given point, the nodal stresses obtained for the various elements are averaged. A 3x3 array of Gauss points was selected.

Since ϕ was represented by u, the x-component of displacement, all other degrees of freedom at each node were fixed.

According to Equations (14) and (15), the elastic material properties of each element were chosen as

$$G_e = 1\ , \qquad E_e = 10^{-5} \qquad (27)$$

Equation (19) is a special case of the general form (1) with

$$g = 2\,G\,\theta\;, \qquad a = b = 0 \tag{28}$$

Hence, from Equation (16), we must apply a "force" to each node given by

$$F_x = 2\,G\,\theta\,G_e\,V \tag{29}$$

(This example shows why it is important to distinguish between G_e, the "shear modulus" specified on the data card for the element, and G, the actual shear modulus for the bar.) This body "force", which is proportional to the volume assigned to each node, is most conveniently specified as a gravitational load, particularly when a consistent, rather than lumped, loading is desired. Since a gravitational field applies the load $\rho_e g_0 V$ to each node, where ρ_e is the element mass density and g_0 is the acceleration due to gravity, it follows that ρ_e and g_0 must be specified so that

$$\rho_e\,g_0 = 2\,G\,\theta\,G_e = 6.4 \tag{30}$$

Thus, since both constants are otherwise arbitrary,

$$\rho_e = 1\;, \qquad g_0 = 6.4 \tag{31}$$

The element thickness is arbitrary since both the "stiffness" matrix and the "load" are proportional to it.

The actual stresses τ_{xz} and τ_{yz} given by Equation (20) may be obtained using the two-dimensional stress-strain law for plane stress elasticity

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & -1 & \\ -1 & 1 & \\ & & 1 \end{bmatrix} \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} \tag{32}$$

where the elastic constants of Equation (27) have been used, and v (the y component of displacement) is everywhere constrained to zero. Hence, since ϕ is represented by u,

$$\begin{cases} \sigma_{xx} = u_{,x} = \phi_{,x} \\ \sigma_{yy} = -u_{,x} = -\phi_{,x} \\ \sigma_{xy} = u_{,y} = \phi_{,y} \end{cases} \tag{33}$$

That is, the stresses designated by the finite element program as σ_{xy} and σ_{yy} (in the global coordinate system) correspond, respectively, to τ_{xz} and τ_{yz}, the shear stresses of interest.

The finite element calculation of the torsional constant using Equation (21) requires the numerical integration of the stress function ϕ over the cross section. For a finite element mesh having N grid points, this integral can be approximated by

$$J = (2/G\theta) \sum_{i=1}^{N} \phi_i A_i \tag{34}$$

where ϕ_i is the value of ϕ (the solution) at point i, and A_i is the area assigned to that point. The area A_i can be easily obtained from the applied "load" vector, since a gravitational field applies a force F_i at point i given by

$$F_i = \rho_e g_0 t A_i \tag{35}$$

where t is the element thickness. Thus,

$$J = (2/G \theta \rho_e g_0 t) \sum_{i=1}^{N} \phi_i F_i \tag{36}$$

where the summation is equal to the dot product of the finite element solution vector and the applied force vector. In NASTRAN, the calculation of J can be made using a simple DMAP ALTER, which is listed in Figure 3. Since, by symmetry, only half of the cross section was modeled, the result obtained using Equation (36) must be doubled to account for the unmodeled half.

The complete NASTRAN data deck used to solve this problem (including the ALTER to calculate the torsional constant) is listed in Figure 3. The coordinate system used for the grid point locations has been translated 30 mm to the left relative to the coordinate system shown in Figures 1 and 4.

The finite element solution thus obtained is compared to the exact solution computed from Equation (24) in Figure 4, which shows a plot of the shear stress τ_{yz} along y = 0 plotted as a function of x. The finite element curve was smoothed slightly by fitting a B-spline curve through the nodal stresses (Ref. 9, 10). The agreement is clearly excellent. The torsional constant obtained by NASTRAN was 252.2 cm^4, which differs by about 0.12% from the exact value of 252.5 cm^4 calculated using Equation (26).

For this particular example, a heat conduction analogy could also be employed. However, for programs like NASTRAN whose heat conduction capability does not allow for convenient specification of uniform heat sources over 2-D elements (the thermal analog of a gravitational field), the user would have the burden of specifying at each point a heat input proportional to the area assigned to that point. For irregular meshes and those modeled with isoparametric elements (for which consistent loads are needed), this burden is substantial.

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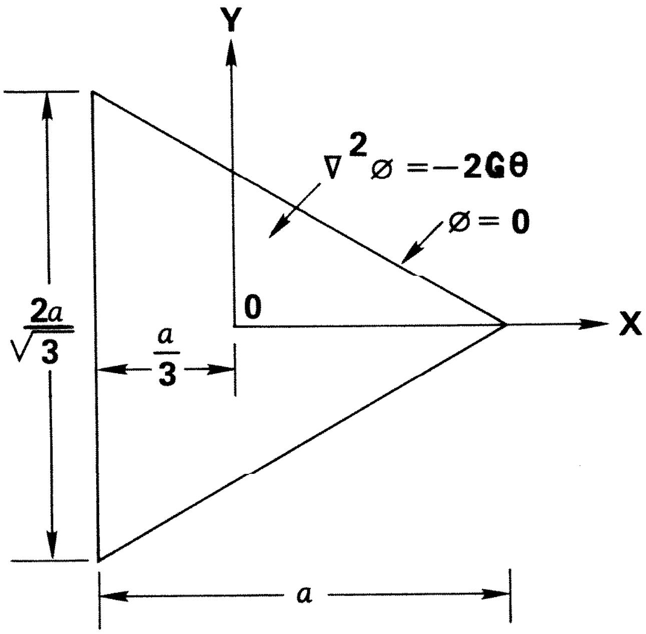

Figure 1 - Torsion of Triangular Prism

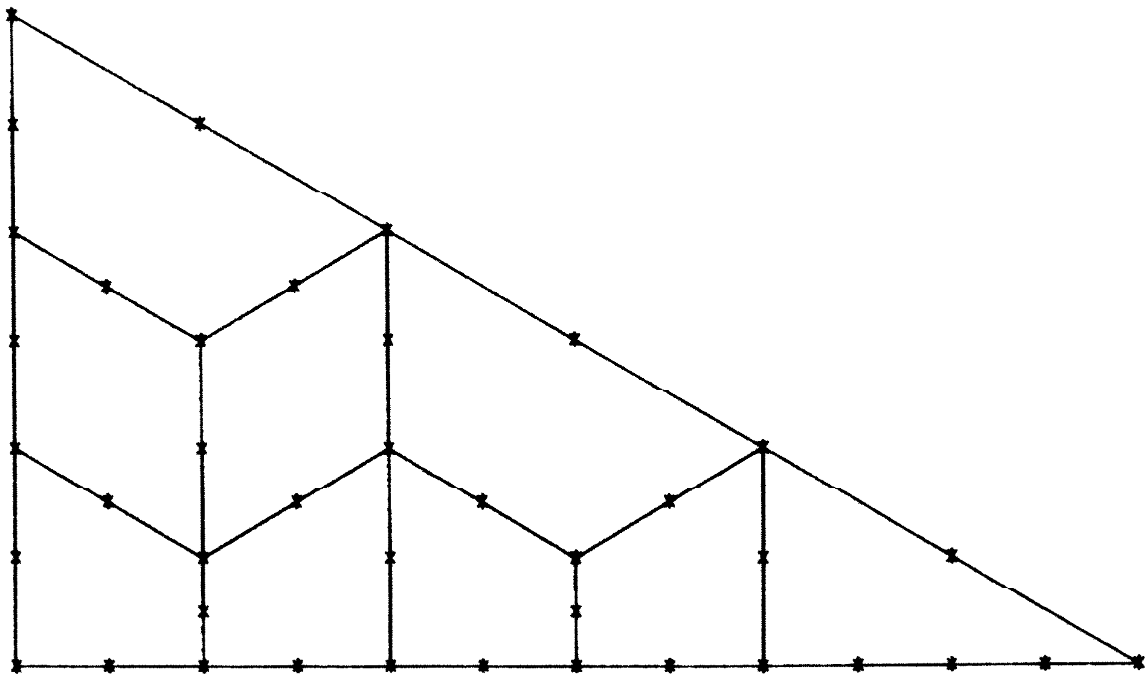

Figure 2 - Finite Element Mesh for Triangular Cross Section

```
ID NASTRAN,DEMO
APP DISP
SOL 1,0
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$     TORSIONAL CONSTANT (TC)
  MPYAD UGV,PG,/TC1/C,N,1/C,N,1/C,N,0/C,N,1 $
  ADD TC1,/TC/C,N,(0.19531,0.0) $   (2/(G*THETA*RHOE*GZERO*T))*2
  MATPRN TC,,,,// $  TORSIONAL CONSTANT
  ALTER 121 $   R.F. 1, NAVY/NASTRAN LEVEL 15
$     STRESS AVERAGING AT GRID POINTS
  STRSAVG EQEXIN,OES1/OESAVG $
  OFP OESAVG,,,,, //V,N,CARDNO $
  SAVE CARDNO $
  ENDALTER $
TIME 10
CEND
TITLE=2-D POISSON EQUATION (TORSION OF TRIANGULAR PRISM)
SPC=13
LOAD=12
DISP=ALL
OLOAD=ALL
STRESS=ALL
BEGIN BULK
APPISO  1
CIS2D8  1       1       3       1       11      13      2       8       +1
+1      12      9       3
CIS2D8  2       1       5       3       13      15      4       9       +2
+2      14      10      3
CIS2D8  3       1       7       5       15      24      6       10      +3
+3      19      16      3
CIS2D8  4       1       13      11      20      22      12      17      +4
+4      21      18      3
CIS2D8  5       1       15      13      22      24      14      18      +5
+5      23      19      3
CIS2D8  6       1       22      20      27      29      21      25      +6
+6      28      26      3
CIS2D8  7       1       24      22      29      35      23      26      +7
+7      32      30      3
CIS2D8  8       1       29      27      33      35      28      31      +8
+8      34      32      3
CIS2D8  9       1       35      33      37      40      34      36      +9
+9      39      38      3
GRAV    12              6.4     1.0
GRDSET                                                  23456
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GRID    29              .045    .00866
GRID    30              .045    .025981
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GRID    32              .0525   .01299
GRID    33              .06
GRID    34              .06     .00866
GRID    35              .06     .017321
GRID    36              .0675
GRID    37              .075
GRID    38              .075    .00866
GRID    39              .0825
GRID    40              .09
MAT7    11      1.E-5   1.0             1.0
PARAM   COUPMASS1
PIS2D8  1       11      1.0
SPC1    13      1       1       THRU    7
SPC1    13      1       16      24      30
SPC1    13      1       35      38      40
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Figure 3 - Listing of NASTRAN Deck

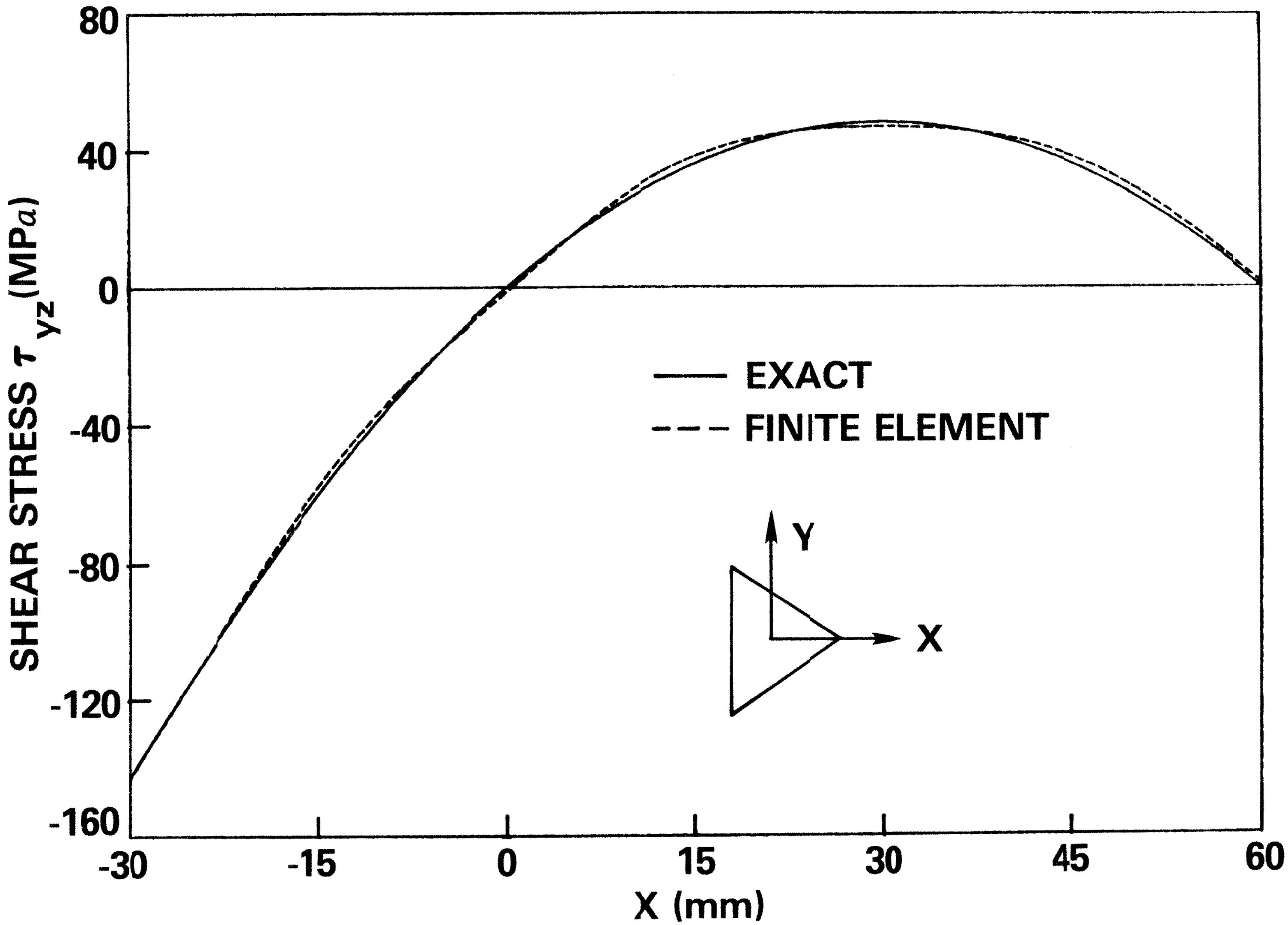

Figure 4 - Shear Stress τ_{yz} Along y = 0: Comparison of Finite Element and Exact Solutions