

DYNAMIC RESPONSE PREDICTION FOR STRUCTURES WITH  
FREQUENCY-DEPENDENT DAMPING TREATMENTS

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ABSTRACT

A finite element procedure is described for calculating the loss factors for elastic structures to which frequency-dependent viscoelastic damping treatments have been applied. The frequency dependence of the viscoelastic damping material is treated by approximating its shear modulus with a second-order polynomial so that the stiffnesses associated with the constant, linear, and quadratic terms can be combined, respectively, with the stiffness, damping, and mass matrices assembled for the rest of the structure. A single complex eigenvalue analysis is then performed in which the eigenvalues are purely imaginary. The loss factor is computed by the modal strain energy (MSE) approach first formulated in 1962 by Ungar and Kerwin and recently recast in the finite element context by Johnson, Kienholz, and Rogers. In the MSE approach, the loss factor of a composite structure vibrating in one of its natural modes may be visualized as a weighted average of the loss factors of the component parts, with the relative stored energies as weighting constants. The finite element procedure, which can treat very general geometries, is illustrated for the case of a vibrating constrained-layer damped plate.

INTRODUCTION

Damping treatments are frequently applied to engineering structures to reduce both vibration and noise. For example, laminated plates composed of alternate layers of elastic and viscoelastic materials have been used as structural members which can dissipate vibratory energy as well as maintain the required structural integrity.

The finite element prediction of the dynamic response of such structures is complicated considerably by the frequency-dependence of the viscoelastic material properties. Because the structure's stiffness matrix is frequency-dependent, the finite element equations are nonlinear rather than linear, and conventional analysis cannot be used. For example, the calculation of time-harmonic forced response by a direct approach (in the physical, rather than modal, coordinates of the structure) would require the complete re-assembly and solution of the system equations for each drive frequency. Also, with most finite element codes, the overall damping characteristics of the structure cannot be determined by solving the complex eigenvalue problem since the frequency-dependence of the coefficient matrices results in a nonlinear eigenvalue problem.

This paper describes how constant coefficient matrices in finite element analysis can be restored so that the difficulties associated with frequency-dependent material properties can be reduced. In essence, a frequency-dependent material property is approximated by a polynomial quadratic in the circular frequency  $\omega$ , and the coefficients of the first and second order terms of the polynomial are combined with the system damping and mass matrices, respectively. The main requirements for this approach are (1) that the frequency-dependence can be adequately represented by a quadratic, and (2) that the finite element code can handle complex stiffness, damping, and mass matrices.

This paper discusses two problems for structures to which frequency-dependent viscoelastic damping treatments have been applied:

1. forced time-harmonic response analysis using physical, rather than modal, coordinates, and
2. prediction of the overall system loss factor (structural damping coefficient).

Convenient solution of the first problem requires only the quadratic polynomial approximation to the frequency dependence of the viscoelastic material properties. For the second problem, the approach taken here is to use the quadratic fit only for the real part of the viscoelastic modulus, and then to solve an undamped (but mathematically complex) eigenvalue problem and apply the modal strain energy (MSE) approach to extract the system loss factor.

#### APPROXIMATION FOR FREQUENCY-DEPENDENT MATERIALS

The formulation of a structural dynamics problem for finite element solution results, in the time domain, in the matrix equation

$$\dot{M}\dot{u} + B\dot{u} + Ku = F(t) \quad (1)$$

where  $M$ ,  $B$ , and  $K$  are the mass, viscous damping, and stiffness matrices, respectively,  $u$  is the vector of unknown displacement components at the grid points, and  $F$  is the vector of applied forces at the grid points.

For time-harmonic loading,

$$F = F_0 e^{i\omega t} \quad (2)$$

$$u = u_0 e^{i\omega t} \quad (3)$$

where  $\omega$  and  $F_0$  are, respectively, the circular frequency and complex amplitude of the applied force, and  $u_0$  is the complex amplitude of the displacement response. In that case, the time-harmonic form of Equation (1) is

$$(-\omega^2 M + i\omega B + K) u_0 = F_0 \quad (4)$$

Consider a structure made of one or more materials, only one of which is damped. Assume that the damped material (which may be anisotropic) is modeled mathematically by the complex modulus approach, in which the modulus of elasticity is the complex number  $E(1+i\eta)$ , where  $\eta$  is the loss factor for the material. Assume also that all moduli for that material have the same loss factor  $\eta$  and, hence, that the Poisson's ratios are real and frequency-

independent. Since every term of the finite element material matrix  $D$  (which converts the strain vector to the stress vector for an element) is proportional to one of the elastic moduli, the frequency dependence can be factored out as a common dimensionless scalar factor:

$$D(\omega) = \alpha(\omega) D_0 (1 + i\eta(\omega)) \quad (5)$$

where  $D_0$  is a real, frequency-independent reference material matrix,  $\alpha(\omega)$  is a dimensionless factor which expresses the frequency dependence of the real part of the elastic moduli, and  $\eta(\omega)$  is the frequency-dependent loss factor for the material. In effect,  $D_0$  is the material matrix for some arbitrary frequency  $\omega_0$ , and  $\alpha D_0$  is the material matrix for some other frequency  $\omega$ ; thus,  $\alpha(\omega_0) = 1$ . It therefore follows that the stiffness matrix  $K_d$  for the portion of the structure made of the damped material is of the form

$$K_d = \alpha K_0 (1 + i\eta) \quad (6)$$

where  $K_0$  is the real part of  $K_d$  at the reference frequency  $\omega_0$ . The stiffness matrix for the remainder of the structure (the undamped portion) will be denoted  $K_e$ .

With damping modeled using the complex modulus approach,  $M$  in Equation (4) is real,  $B = 0$  (unless viscous damping is also present), and  $K$  is complex. The real part of  $K$  includes the contributions from all components an materials in the structure. However, if only one material has complex moduli, the imaginary part of the system stiffness matrix  $K$  arises only from the damped material, thus allowing the stiffness matrix for the damping material to be extracted from the overall system stiffness matrix. If the loss factor (structural damping coefficient)  $\eta$  for the damping material is specified as unity on the finite element material properties data card,

$$\alpha K_0 = \text{Im}(K_d) |_{\eta=1} = \text{Im}(K) |_{\eta=1} \quad (7)$$

The stiffness matrix  $K_e$  for the remainder of the structure (the undamped portion) is then obtained by subtracting from the system matrix  $K$  the contribution from the damping material:

$$K_e = \text{Re}(K) - \text{Im}(K) |_{\eta=1} \quad (8)$$

where  $K$  is the (complex) stiffness matrix for the entire structure assuming that  $\eta = 1$  is specified for the damped portion.

It will be convenient to write Equation (6) in the form

$$K_d = K_0(\alpha + i\beta) \quad (9)$$

where

$$\beta(\omega) = \alpha(\omega)\eta(\omega) \quad (10)$$

If the dependence of  $\alpha$  and  $\beta$  on  $\omega$  were quadratic, it is clear from Equation (4) that the frequency-dependent terms could be absorbed into the system's mass and viscous damping matrices, thereby transforming the problem into standard form (although with complex coefficient matrices).

Therefore, we attempt to approximate  $\alpha(\omega)$  and  $\beta(\omega)$  with the quadratic polynomials

$$\alpha(\omega) \cong \alpha_0 + \alpha_1\omega + \alpha_2\omega^2 \quad (11)$$

$$\beta(\omega) \cong \beta_0 + \beta_1\omega + \beta_2\omega^2 \quad (12)$$

The six unknown coefficients in Equations (11) and (12) can be determined by standard least-squares procedures (which are summarized in the next section).

With quadratic frequency-dependence of the damping material, the coefficient matrices  $M$ ,  $B$ , and  $K$  in Equation (4) can be replaced by new complex matrices  $M^*$ ,  $B^*$ , and  $K^*$  defined by the equality

$$-\omega^2 M + i\omega B + K_e + K_0[\alpha_0 + \alpha_1\omega + \alpha_2\omega^2 + i(\beta_0 + \beta_1\omega + \beta_2\omega^2)] = -\omega^2 M^* + i\omega B^* + K^* \quad (13)$$

where  $K_e$  is the system stiffness matrix for the undamped part of the structure,  $B$  is the damping matrix for the viscous dampers, if any, and  $M$  is the original mass matrix for the entire structure. Thus, by equating coefficients of like powers of  $\omega$  in Equation (13), we obtain

$$K^* = K_e + (\alpha_0 + i\beta_0)K_0 \quad (14)$$

$$B^* = B + (\beta_1 - i\alpha_1)K_0 \quad (15)$$

$$M^* = M + (-\alpha_2 - i\beta_2)K_0 \quad (16)$$

where  $K_e$  is determined from Equation (8). In this form, the new coefficient matrices  $M^*$ ,  $B^*$ , and  $K^*$  are complex and independent of frequency.

To summarize, the frequency dependence of the real and imaginary parts of the moduli of the viscoelastic material is replaced by a quadratic so that such dependence can be absorbed into the system mass and damping matrices. The principal assumptions made are (1) that the frequency dependence of the material can be described by a single scalar function, (2) that this function can be adequately represented by a quadratic polynomial, and (3) that the finite element code is general enough to allow complex coefficient matrices.

#### CURVE-FITTING

Let  $y(x)$  represent one of the two frequency-dependent material property functions  $\alpha(\omega)$  and  $\beta(\omega)$  defined in Equation (9). Assume that  $y(x)$  is known at  $n$  points  $(x_i, y_i)$ ,  $i=1, 2, \dots, n$ . We wish to approximate  $y(x)$  by the quadratic polynomial

$$p(x) = a_0 + a_1x + a_2x^2 \quad (17)$$

in such a way that the residual

$$R = \sum_{i=1}^n w_i [y_i - (a_0 + a_1x_i + a_2x_i^2)]^2 \quad (18)$$

is minimized, where  $w_i$  is the weighting factor for point  $i$ . This is the classical least squares problem. To minimize the squares of the absolute errors,  $w_i = 1$ . To minimize the squares of the relative errors,  $w_i = 1/y_i^2$ .

The polynomial coefficients which minimize the residual  $R$  are the solutions of the symmetric system [1]

$$\begin{bmatrix} \sum w_i & \sum w_i x_i & \sum w_i x_i^2 \\ \sum w_i x_i & \sum w_i x_i^2 & \sum w_i x_i^3 \\ \sum w_i x_i^2 & \sum w_i x_i^3 & \sum w_i x_i^4 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sum w_i y_i \\ \sum w_i x_i y_i \\ \sum w_i x_i^2 y_i \end{pmatrix} \quad (19)$$

where all summations are from  $i=1$  to  $n$ .

#### MODAL STRAIN ENERGY APPROACH TO DAMPING

The relationship between damping and energy concepts for nonhomogeneous structures was apparently first formulated by Ungar and Kerwin in 1962 [2]. In general terms, they showed that the loss factor of a composite structure may be computed as a weighted average of the loss factors of the component parts, with the relative stored energies as weighting factors. The Ungar-Kerwin ideas were recently recast in the finite element context by Johnson, Kienholz, and Rogers [3-5]. For completeness, we summarize (in slightly different form) the aspects of this work needed here.

The complex Rayleigh quotient for a damped structure may be written in the form

$$\omega^2(1 + i\eta) = \phi_d^T (K_T + i\eta_d K_d) \phi_d / (\phi_d^T M \phi_d) \quad (20)$$

where  $K_T$  = stiffness matrix (real) for the entire structure

$K_d$  = stiffness matrix (real) for the part of the structure which is damped (i.e., the viscoelastic material)

$\eta_d$  = loss factor (real) for the viscoelastic material

$M$  = mass matrix (real) for the entire structure

$\phi_d$  = complex eigenvector (mode shape) for a damped mode of the structure

$\omega$  = circular natural frequency (real) for the mode

$\eta$  = composite loss factor (real) for the entire structure

For lightly damped structures, the damped mode shape  $\phi_d$  may be approximated by the real eigenvector  $\phi$  obtained by solving the real, undamped eigenvalue problem.

Using this approximation ( $\phi_d \approx \phi$ ) and equating the imaginary parts of Equation (20) to each other yields

$$\eta = \eta_d (\phi^T K_d \phi) / (\omega^2 \phi^T M \phi) \quad (21)$$

Since the real eigenvector  $\phi$  satisfies

$$\omega^2 \phi^T M \phi = \phi^T K_T \phi \quad (22)$$

we obtain

$$\eta = \eta_d(\phi^T K_d \phi) / (\phi^T K_T \phi) \quad (23)$$

or

$$\eta = \eta_d(k_d/k_T) \quad (24)$$

where  $k_T$  is the generalized stiffness of the mode, and  $k_d$  is the contribution of the viscoelastic material to the generalized stiffness. Since the generalized stiffness for a mode is equal to twice the elastic strain energy in the mode, Equation (24) may be interpreted as stating that the ratio of the composite loss factor  $\eta$  to the loss factor of the viscoelastic material is equal to the fraction of the total strain energy contained in the damping materials.

More generally, for structures containing several damping materials with individual loss factors  $\eta_1, \eta_2, \eta_3 \dots$ , the composite loss factor for the structure is

$$\eta = \eta_1(k_1/k_T) + \eta_2(k_2/k_T) + \dots \quad (25)$$

where  $k_i$  is the contribution of the  $i$ th damping material to the total generalized stiffness  $k_T$ .

Equation (24) is implemented within the finite element procedure by performing an undamped eigenvalue analysis and extracting for each mode the total strain energy  $2k_T$  and the strain energy  $2k_d$  contained in the viscoelastic material. In the undamped analysis, the damping material is modeled and assigned a zero loss factor.

The details of the implementation of these procedures specifically for NASTRAN were presented recently by Everstine and Marcus [6].

#### EXAMPLE

The procedure for calculating composite loss factors will be illustrated for the three-layer sandwich plate shown in Figure 1.

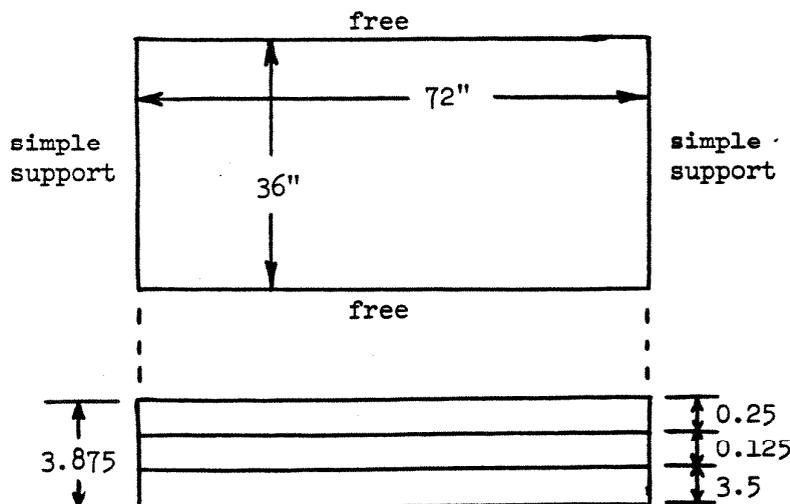


Figure 1 - Three-Layer Sandwich Plate Used for Calculating Loss Factors

We assume that the shear modulus  $G$  and loss factor  $\eta$  for the middle layer are given by

$$G(f) = 10^4 + 9f \quad (\text{psi}) \quad (26)$$

$$\eta(f) = 0.5 - (4 \times 10^{-5})f \quad (27)$$

where  $f$  is frequency in Hz. These relations imply considerable frequency dependence, since between zero and 10,000 Hz,  $G$  varies by a factor of ten, and  $\eta$  varies by a factor of five. The top and bottom layers are made of steel.

The plate was modeled with three layers of 20-node isoparametric ("brick") finite elements. A  $13 \times 5 \times 3$  mesh of elements was used. These elements were used rather than plate elements because of interest in fairly thick plates of variable thickness, in which case the engineering plate theory no longer applies. Guyan reduction was applied to this model in such a way that the only degrees of freedom retained were the normal translations at the corner nodes of each element on the top and bottom faces.

The composite loss factor computed using the finite element procedure described in the preceding sections is compared in Figure 2 with a calculation based on classical infinite plate theory presented by Ross, Ungar, and Kerwin [7]. That figure indicates a good agreement between the

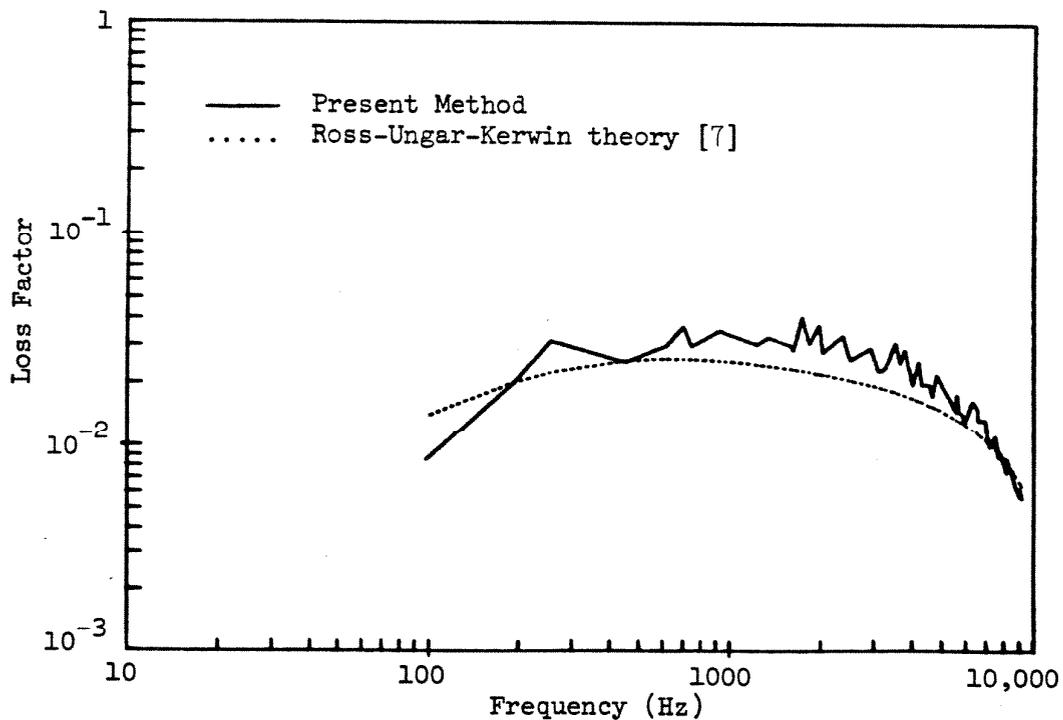


Figure 2 - Comparison of Loss Factors for Sandwich Plate as Predicted by Present Finite Element Method and by Ross-Ungar-Kerwin Theory [7].

two predictions. The infinite plate theory is generally thought to provide a reasonably accurate prediction of the composite loss factor for plates with this simple geometry, and hence serves as a good check on the finite element result.

#### DISCUSSION

A finite element procedure has been described for predicting the loss factors of structures to which frequency-dependent viscoelastic damping treatments have been applied. This procedure, used on the simple test problem of a three-layer plate, yielded predictions similar to those of a generally-accepted analytical procedure for infinite plates.

The power of the finite element procedure, however, is that it is not restricted to such simple geometry but can readily handle such complications as different boundary conditions, variable plate thickness, and localized treatments. None of these effects can be treated by the infinite plate theory.

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