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3.3 SYMMETRY

The finite-element analysis of engineering problems with complex geometry is tedious at best. However, for problems with symmetry, it is possible to gain some information about the solutions from the symmetry alone [35-41]. Moreover, with symmetry present, the analyst needs to model only a portion of the overall region of interest, thereby saving both the analyst's time and the computer's time, the former generally being the more valuable. For example, a structure possessing one plane of mirror symmetry can be analyzed by modeling only one-half the structure, whether the loads are symmetric or not. Nonsymmetric loads, for which case the half structure must be analyzed twice, still have benefits, since a half structure generally costs much less than half as much to analyze as the complete structure would cost. For a half structure, the total number of degrees of freedom N in the model is about half that of an equivalent full model, and the model's matrix bandwidth B (either maximum or average) is also usually reduced, sometimes by as much as 50%. The overall computer solution involves equation solving (for which the computer time is proportional to NB^2) and numerous other operations (for which the time is roughly proportional to N) [2].

To be useful, symmetry must be exploited systematically and with confidence. This section reviews and summarizes the basic concepts involved in the systematic application of symmetry in finite-element analysis, where *symmetry* refers to objects rather than physical laws [42] or materials. In general, group theory is the mathematical language for discussions of symmetry (particularly in quantum mechanics), but such an approach is not necessary for finite-element applications and will not be used here. This discussion will, at times, emphasize structural applications since most finite-element analyses are in that area; however, the same concepts carry over into other areas.

3.3A Types of Symmetry and Definitions

In engineering applications, the most commonly encountered types of symmetry are reflective (or mirror) symmetry, rotational (or axial) symmetry, and inversion symmetry [39]. Examples of these three types are shown in Fig. 3.30.

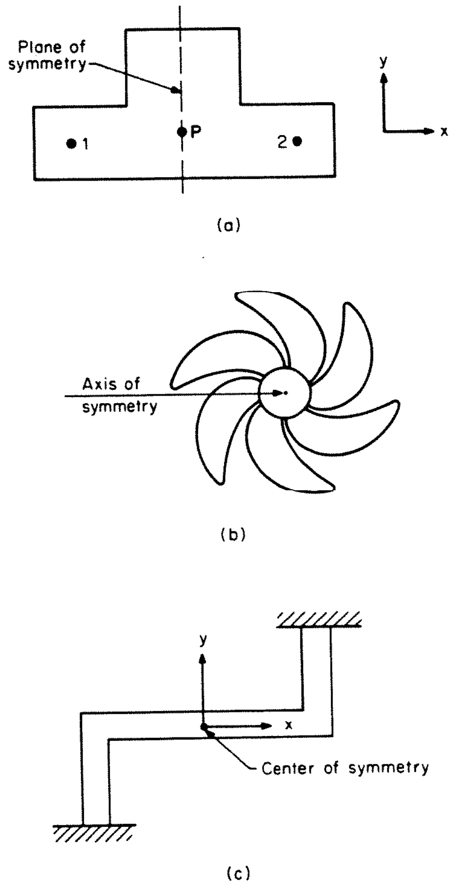

FIG. 3.30 Examples of different types of symmetry: (*a*) Reflective symmetry. (*b*) Rotational symmetry. (*c*) Inversion symmetry.

An object possesses symmetry if the application to the object of some operation (such as a reflection, rotation, or inversion) transforms the object into an equivalent configuration. For engineering applications, this characterization of symmetry requires not only geometrical symmetry, but also symmetry with respect to material properties and restraints. For example, in Fig. 3.30*b*, if one propeller blade were made of aluminum and another of bronze (an unlikely situation), there would be no symmetry to exploit. In some situations, other properties may also play a role in deciding the presence of symmetry; for example, thermal radiation problems require symmetry with respect to color. Symmetry can normally be identified by inspection.

The characterization of symmetry in a particular situation is not necessarily unique. For example, the symmetry of Fig. 3.30*c* can also be characterized as a sequence of two reflections, one in the *yz*-plane followed by one in the *xz*-plane, or vice versa. The same structure could also serve as an example of a structure with rotational symmetry (with a rotation angle of 180°).

In general, reflective symmetry is viewed as the fundamental type of symmetry, since it can be shown that all symmetric transformations of finite figures in three dimensions reduce to successive reflections in not more than three planes (which might not even be planes of symmetry) [43].

Once the symmetry properties of a structure are identified, the loads can be addressed. The question of whether a given system of loads is symmetric depends on the structure to which that system is applied. Specifically, a system of loads, when applied to a structure possessing certain symmetry, is defined as *symmetric* if it is transformed into an equivalent configuration by the symmetry operations of the structure [39]. The system of loads is defined as *antisymmetric* if the symmetry operations plus a negation of signs of all loads transform them into an equivalent configuration. For example, the loads in Fig. 3.31 are symmetric when applied to that structure, whose symmetry is characterized by the sequence of two reflections indicated above.

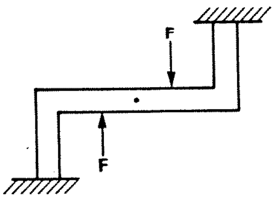

FIG. 3.31 A structure with symmetric loads.

3.3B The Guiding Principle

The principle on which all applications of symmetry are based is that "equivalent causes produce equivalent effects," or, more generally, "the effect is at least as symmetric as the cause" [35, 39]. In the context of structural mechanics, the practical effect of this principle is that symmetric loads produce symmetric effects (displacements, stresses, etc.) and antisymmetric loads produce antisymmetric effects.

3.3C Boundary Conditions

When only a portion of a symmetric structure is modeled, the basic principle provides the tool for systematically deriving the symmetric or antisymmetric boundary conditions which must be applied at artificial boundaries introduced because of symmetry. Emphasis in this section will be restricted to planes of symmetry, since reflective symmetry has already been identified as the fundamental type of symmetry.

Consider, for example, the symmetric region shown in Fig. 3.30*a*, where *P* denotes a typical point in the plane of symmetry. Define a cartesian coordinate system with the *x*-direction normal to the plane of symmetry and the *yz*-plane parallel to the plane of symmetry. To derive the symmetric and antisymmetric boundary conditions to be applied at *P* if only half the region is modeled, (1) consider in turn each displacement component at that point, (2) apply to that component (assumed to be nonzero) the symmetry (or antisymmetry) operations characterizing the structure, and (3) observe whether or not the component is transformed into itself. If it is not transformed into itself, the component must vanish in order not to violate symmetry. The relevant symmetry operation is a reflection onto the *yz*-plane containing point *P*. The antisymmetry operations consist of the same reflection followed by a negation of sign. For example, assume that u_x, the *x*-component of displacement, is nonzero at *P*. The reflection produces an image of u_x with the opposite orientation. The additional negation of sign (for antisymmetry) yields a result coinciding with the original configuration. Therefore, u_x must vanish at *P* in order not to violate symmetry, but u_x may be nonzero for antisymmetric behavior. Similarly, we find that u_y and u_z vanish at *P* for antisymmetric behavior and may be nonzero for symmetric behavior.

Rotational degrees of freedom, if present, require slightly different treatment. Let points 1 and 2 in Fig. 3.30*a* be image points of each other. Symmetric moments applied to these points must have opposite signs. For example, if the moment $M_z = 10\ \text{lb} \cdot \text{in}$ is applied at point 2, its symmetric counterpart is the moment $M_z = -10\ \text{lb} \cdot \text{in}$ at point 1. In other words, the reflection of an axial vector (a rotation or moment) onto a plane requires an additional negation of sign compared with the way true vectors reflect [39]. The mathematical basis for this result is that reflection is an improper orthogonal transformation.

The application of the symmetry operation (reflection) to the rotational components R_x, R_y, and R_z at point *P* in Fig. 3.30*a* indicates that for symmetric behavior, R_y and R_z must vanish in order not to violate symmetry, and $R_x = 0$ for antisymmetry. To summarize, the boundary conditions to impose at point *P* are

$$\begin{aligned} u_x = R_y = R_z = 0 \qquad & \text{for symmetry} \\ R_x = u_y = u_z = 0 \qquad & \text{for antisymmetry} \end{aligned} \tag{3.18}$$

The results expressed in these equations may be generalized as follows: Points lying in a plane of *symmetry* can suffer no translation out of the plane and no rotation about in-plane lines. For *antisymmetry* the complementary set of degrees of freedom is constrained. The complementary nature of the symmetric and antisymmetric boundary conditions is a general result which follows from the observation that the only distinction between antisymmetry and symmetry is the additional negation in the symmetry operations.

When higher-order derivatives are used as degrees of freedom, additional symmetric and antisymmetric constraints are also required. Such constraints are most easily derived from the observations that u_x, the translational component of displacement normal to a symmetry plane, is either an odd or an even function of *x*, depending on whether the behavior is symmetric or antisymmetric, respectively. Similarly, the translational components of displacement parallel to a symmetric plane are even and odd in *x* for symmetric and antisymmetric behavior, respectively [44]. For even functions of *x*, the odd-order derivatives with respect to *x* must vanish for symmetry, and the even-order derivatives with respect to *x* vanish for antisymmetry. Conversely, for odd functions of x, the even- and odd-order derivatives with respect to x vanish for symmetry and antisymmetry, respectively. As before, the set of degrees of freedom constrained as a consequence of symmetry is complementary to the set which is constrained as a consequence of antisymmetry; i.e., whichever freedoms are constrained for symmetry will be free for antisymmetry, and vice versa. This follows directly from the definitions, since the only real distinction between symmetry and antisymmetry is the additional negation that distinguishes even from odd functions. Thus, if one can determine for some problem the (correct) symmetric boundary conditions, the antisymmetric conditions consist of the complementary set of degrees of freedom. One consequence of this property is that the total number of degrees of freedom arising from symmetric and antisymmetric models of a structure must equal the original number of degrees of freedom for the entire structure, disregarding symmetry.

The symmetric conditions for scalar field problems (e.g., heat conduction or potential fluid flow) are obtained as special cases of the preceding development. At a plane of symmetry, the normal derivative of the field variable vanishes; this is a natural boundary condition in finite-element analysis. For antisymmetry, the function itself must vanish at a plane of symmetry.

3.3D Nonsymmetric Loads

In general, most load systems applied to structures are neither symmetric nor antisymmetric, but nonsymmetric. However, any nonsymmetric system can always be uniquely decomposed into the sum of a symmetric and an antisymmetric system of loads [45, 46], as, for example,

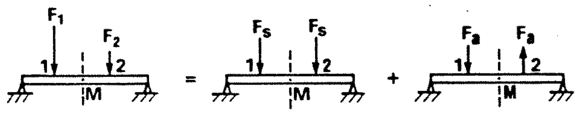

FIG. 3.32 Decomposition of nonsymmetric loads.

in Fig. 3.32. Given arbitrary loads F_1 and F_2 at a point 1 and its image, point 2, the symmetric part of the load F_s and the antisymmetric part F_a are given by

$$F_s = \frac{F_1 + F_2}{2}$$
$$F_a = \frac{F_1 - F_2}{2} \qquad (3.19)$$

In linear problems, to which the principle of superposition applies, only half the problem shown in Fig. 3.32 needs to be modeled. The analyst would model the left half, say, and solve the problem in two steps: (1) the symmetric part of the load is applied along with symmetric boundary conditions at the middle M, and (2) the antisymmetric part of the load is applied along with antisymmetric boundary conditions imposed at M. Thus, we have Fig. 3.33*a*, for which the symmetric (S) and antisymmetric (A) boundary conditions are given in Eq. 3.18. Adding the two solutions in Fig. 3.33*a* yields the solution of the original problem

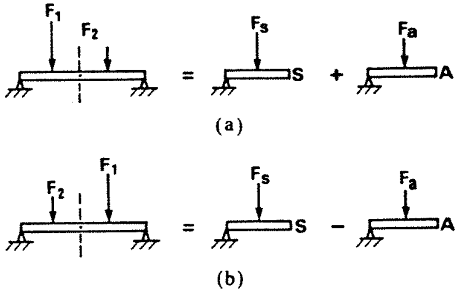

FIG. 3.33 Superposition of symmetric and antisymmetric solutions. (*a*) Modeled side. (*b*) Unmodeled side.

only for the left side of the structure (the modeled side). To obtain the solution for the right side (the unmodeled side), the two solutions can be subtracted, as indicated in Fig. 3.33*b*. Taking the difference of the symmetric and antisymmetric solutions has the practical effect of reversing the role played by the left and right sides. Thus, even though only the left side is modeled, the entire solution can be obtained.

3.3E Multiple Planes of Symmetry

The rectangular region shown in Fig. 3.34 possesses two planes of symmetry (*xz* and *yz*); hence the problem can be decomposed into four parts, as shown. Any quadrant can be chosen to be modeled, and the four combinations of symmetric and antisymmetric boundary conditions (*SS*, *SA*, *AS*, and *AA*) imposed on the points lying in the two planes of symmetry. The four solutions can be combined in various ways to yield the solutions in all four quadrants [41, 46].

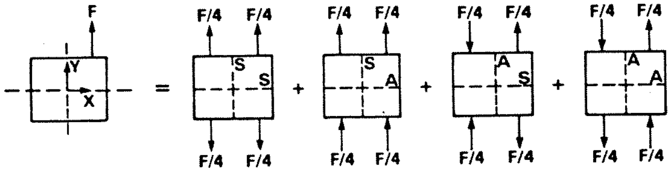

FIG. 3.34 Multiple planes of symmetry.

3.3F Free Vibrations

The foregoing discussion has been devoted exclusively to statics problems, but free-vibration problems (eigenvalue problems) can also exploit symmetry. The calculation of all natural frequencies and mode shapes of a symmetric structure would require one eigenvalue analysis for each unique combination of symmetric and antisymmetric boundary conditions. For example, the natural frequencies of the region of Fig. 3.34, which has two orthogonal planes of symmetry, can be obtained by modeling only one quadrant and applying, in turn, each of the four combinations of boundary conditions.

The total number of degrees of freedom involved in the four component problems of Fig. 3.34 exactly equals the original number of DOF contained in the complete problem [44, 47]. This follows as a direct consequence of the symmetric and antisymmetric boundary conditions involving complementary sets of DOF. Thus we have "conservation of DOF." If this were not so, we would have the disturbing situation in which the mere application of symmetry would result in the creation or destruction of DOF. The purpose of applying symmetry is, of course, to solve a problem with less effort rather than to create a different one.

3.3G Time-Dependent Problems

All the preceding results for statics problems also apply to linear transient (time-dependent) situations, except that the entire history of time-dependent loads must be decomposed into symmetric and antisymmetric parts. This decomposition has been illustrated in the context of underwater shock response, for example, by Everstine [48].

3.3H Finite Elements in Planes of Symmetry

Special consideration is necessary to treat the situation in which elements lie entirely in the plane of symmetry (i.e., the grid points which define the element lie entirely in the plane). For example, in the stiffened plate shown in Fig. 3.35, the beam stiffener (modeled with beam

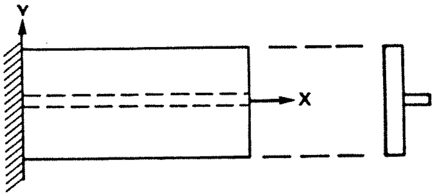

FIG. 3.35 An example of a symmetry plane containing a finite element.

elements) lies entirely in the xz-plane, which is a plane of structural symmetry. Although the symmetric boundary conditions are unaffected by this situation, care must be exercised in computing the geometrical properties of a beam element lying in the plane of symmetry. In particular, the properties for each "half element" should be specified so that the half element receives one-half the total *stiffness* rather than one-half the cross section. For example, properties such as area A, cross-sectional moments of inertia I_1, I_2, and I_{12}, and torsional constant J are first computed for the full cross section before entering one-half of those values. (Note that J and either I_1 or I_2 do not depend linearly on individual cross-sectional dimensions.)

To prove the validity of this approach, we need only treat each half the symmetric structure as a "superelement" involving many grid points. Then, if the two sides are recombined using the usual rules of matrix assembly, the resulting stiffness matrix would have to be the correct stiffness matrix for the entire structure. Thus, when an element is cut in half by a plane of symmetry, each side receives one-half the total stiffness.

3.3i Cyclic Symmetry

Many structures, including turbine and pump impellers, rotating machines, space antennas, and propellers, exhibit rotational symmetry such as that shown in Fig. 3.30*b*. Such structures are made up of identical segments arranged symmetrically with respect to an axis. Because of symmetry, the only portion of the structure that must be modeled and analyzed is the smallest repeating segment.

Although several contributions have been made to various aspects of the problem [6, 36, 49–57], the most general treatments are those of Hussey [54], MacNeal, Harder, and Mason [6], and Thomas [55]. In the work by MacNeal, Harder, and Mason, an automated procedure was developed for both simple rotational symmetry and dihedral symmetry, a special case in which each segment has a plane of reflective symmetry. The theoretical approach, which is too lengthy to be repeated here, is based on a finite Fourier series transformation in the azimuthal coordinate. Nonsymmetric loads can also be handled. Irons and Ahmad include a good discussion of cyclic symmetry (referred to as *sectorial symmetry*) [58].

A special case of cyclic symmetry occurs with axially symmetric structures (bodies of revolution) for which the geometry is completely independent of the azimuthal coordinate. Nonsymmetric loads for such situations are often handled for linear problems by expanding the loads and all solution variables in a truncated Fourier series in the azimuthal coordinate.

3.5 REFERENCES

2. Zienkiewicz, O. C., *The Finite Element Method*, 3d ed., McGraw-Hill, New York, 1977.

6. MacNeal, R. H., R. L. Harder, and J. B. Mason, "NASTRAN Cyclic Symmetry Capability," *NASTRAN: Users' Experiences*, NASA TM X-2893, National Aeronautics and Space Administration, Washington, D.C., 1973.

35. Rosen, J., *Symmetry Discovered*, Cambridge University Press, Cambridge, 1975.
36. Miller, A. G., "Application of Group Representation Theory to Symmetric Structures," *Appl. Math. Model.*, **5**:290–294 (1981).
37. Rosen, J., "Symmetry: An Introduction to Its Theory and Application in Physics (A Resource Article for Teachers)," Rep. TAUP-307-72, Department of Physics and Astronomy, Tel-Aviv University, Tel-Aviv, 1972.
38. Rosen, J., *A Symmetry Primer for Scientists*, Wiley, New York, 1982.
39. Glockner, P. G., "Symmetry in Structural Mechanics," *J. Struct. Div. ASCE*, **99**(ST1):71–89 (1973); discussion by K. R. Leimbach and D. Franz, **99**(ST8):1792–1794 (1973).
40. Renton, J. D., "On the Stability Analysis of Symmetrical Frameworks," *Q. J. Mech. Appl. Math.*, **17**(2):175–197 (1964).
41. Kardestuncer, H., and K. Berg. "Matrix Analysis of Large Symmetric Skeletal Systems," in P. G. Glockner and M. C. Singh (eds.), *Symmetry, Singularity and Group Theoretic Methods in Mechanics*, Calgary, Alta., 1974.
42. Smith, C. L., "Symmetry and the Laws of Nature," *New Sci.*, **92**(1274):94–97 (Oct. 8, 1981).
43. Shubnikov, A. V., and V. A. Koptik, *Symmetry in Science and Art*, Plenum Press, New York, 1974.
44. Everstine, G. C., "Comment on 'On Symmetries and Antisymmetries in Solving Vibration Problems Using High Precision Finite Elements,'" *J. Sound Vibration*, **52**(1):143–144 (1977).
45. Newell, J. S., "Symmetric and Anti-Symmetric Loadings," *Civ. Eng.* **9**(4):249–251 (1939).
46. Butler, T. G., "Using NASTRAN to Solve Symmetric Structures with Nonsymmetric Loads," *Tenth NASTRAN Users' Colloquium*, NASA CP-2249, National Aeronautics and Space Administration, Washington, D.C., 1982, pp. 216–232.

47. Everstine, G. C., "The Application of Structural Symmetry in Finite Element Analysis," TM-184-77-05, David Taylor Naval Ship Research and Development Center, Bethesda, Md., 1977.

48. Everstine, G. C., "A NASTRAN Implementation of the Doubly Asymptotic Approximation for Underwater Shock Response," *NASTRAN: Users' Experiences*, NASA TM X-3428, National Aeronautics and Space Adminstration, Washington, D.C., 1976, pp. 207–228.

49. Zienkiewicz, O. C., and F. C. Scott, "On the Principle of Repeatability and Its Application in Analysis of Turbine and Pump Impellers," *Int. J. Numer. Meth. Eng.*, **4**(3):445–448 (1972).

50. Noor, A. K., and R. A. Camin, "Symmetry Considerations for Anisotropic Shells," *Comput. Meth. Appl. Mech. Eng.*, **9**:317–335 (1976).

51. Noor, A. K., M. D. Mathers, and M. S. Anderson, "Exploiting Symmetries for Efficient Postbuckling Analysis of Composite Plates," *AIAA J.*, **15**(1):24–32 (1977).

52. Mangalgiri, P. D., B. Dattaguru, and T. S. Ramamurthy, "Specification of Skew Conditions in Finite Element Formulation," *Int. J. Numer. Meth. Eng.*, **12**(6):1037–1041 (1978).

53. Evensen, D. A., "Vibration Analysis of Multi-Symmetric Structures," *AIAA J.*, **14**(4):446–453 (1976).

54. Hussey, M. J. L., "General Theory of Cyclically Symmetric Frames," *J. Struct. Div. ASCE*, **93**(ST2):163–176 (1967).

55. Thomas, D. L., "Dynamics of Rotationally Periodic Structures," *Int. J. Numer. Meth. Eng.*, **14**(1):81–102 (1979).

56. Melvin, M. A., and S. Edwards, Jr., "Group Theory of Vibrations of Symmetric Molecules, Membranes, and Plates," *J. Acoust. Soc. Am.*, **28**(2):201–216 (1956).

57. Zheng, X., G. Bao, and S. Sun, "Applications of Group Theory to Vibrational Analysis of Shell Structure with Space Rotation Symmetry," in H. Guangqian and Y. K. Cheung (eds.), *Proceedings of the International Conference on Finite Element Methods*, Science Press, Beijing, China, Gordon and Breach, New York, 1982.

58. Irons, B., and S. Ahmad, *Techniques of Finite Elements*, Ellis Horwood, Wiley, Chichester, 1980.