Elasticity

by

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Preface

These lecture notes are intended to supplement a one-semester graduate-level engineering course at The George Washington University in the theory of elasticity. Although the emphasis is on the Cartesian tensor approach, the direct (vector-operator) approach is also used where appropriate. The main prerequisites are elementary mechanics of materials, a standard calculus sequence, and some exposure to linear algebra and matrices. In general, the mix of topics and level of presentation are aimed at graduate students in civil, aerospace, and mechanical engineering.

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1 Mathematical Preliminaries

1.1 Vectors

We denote vector quantities in print using boldface letters (e.g., $\mathbf{A}$) and, when hand-written, with an underline (e.g., $\underline{A}$). The basis vectors in Cartesian coordinates are denoted $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in the x_1, x_2, and x_3 directions, respectively. We prefer the use of numerical subscripts for the vector components and basis vectors, since the equations of elasticity can be developed in a very compact form using index notation and Cartesian tensors. Thus, in Cartesian coordinates, the vector $\mathbf{x}$ can be written

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \tag{1.1}$$

where x_1, x_2, and x_3 are the Cartesian components of $\mathbf{x}$.

Note that the basis vectors are unit vectors (vectors with unit length), so that

$$|\mathbf{e}_1| = |\mathbf{e}_2| = |\mathbf{e}_3| = 1 \quad \text{or} \quad |\mathbf{e}_i| = 1, \quad i = 1, 2, 3. \tag{1.2}$$

The basis vectors are also mutually perpendicular:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = \mathbf{e}_2 \cdot \mathbf{e}_3 = \mathbf{e}_3 \cdot \mathbf{e}_1 = 0. \tag{1.3}$$

In general,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \tag{1.4}$$

where δ_{ij} is the *Kronecker delta* defined as

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{1.5}$$

Thus, the three basis vectors form an *orthonormal basis* (a basis whose basis vectors are mutually orthogonal unit vectors).

We can also write $\mathbf{x}$ using the *summation convention*, where twice-repeated indices are summed over the range (1, 2, 3 in three dimensions), and simply write

$$\mathbf{x} = x_i\mathbf{e}_i. \tag{1.6}$$

Here the summation over the range $1, 2, 3$ is implied, since the dummy index i appears twice.

With the summation convention,

$$\delta_{ii} = \begin{cases} \delta_{11} + \delta_{22} = 2 \text{ in 2-D} \\ \delta_{11} + \delta_{22} + \delta_{33} = 3 \text{ in 3-D.} \end{cases} \tag{1.7}$$

Since the Kronecker delta is the index notation form of the identity matrix $\mathbf{I}$, δ_{ii} is the *trace* of $\mathbf{I}$ (the sum of the diagonal entries).

Since the unit basis vectors form a right-handed orthogonal triad,

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2, \tag{1.8}$$

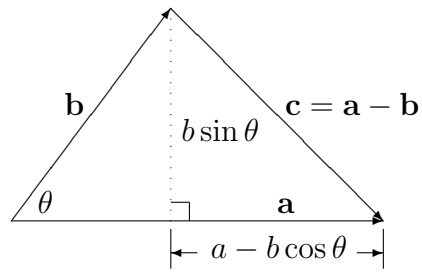


Figure 1: Two Vectors.

$$\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3, \quad \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1, \quad \mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2. \tag{1.9}$$

These relations can be summarized using the single equation

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k, \tag{1.10}$$

where the dummy index k is summed, and e_{ijk} is the *alternating symbol* (or *permutation symbol*) defined as

$$e_{ijk} = \begin{cases} +1, & \text{if the subscripts form an even permutation of 123,} \\ -1, & \text{if the subscripts form an odd permutation of 123,} \\ 0, & \text{otherwise (e.g., if two subscripts are equal).} \end{cases} \tag{1.11}$$

For example,

$$\begin{aligned} e_{123} &= e_{231} = e_{312} = 1, \\ e_{213} &= e_{321} = e_{132} = -1, \\ e_{113} &= e_{133} = e_{221} = 0. \end{aligned} \tag{1.12}$$

Consider two vectors $\mathbf{a}$ and $\mathbf{b}$ (Fig. 1) given by

$$\mathbf{a} = a_i\mathbf{e}_i, \quad \mathbf{b} = b_i\mathbf{e}_i. \tag{1.13}$$

Since the index i is a dummy index of summation, any symbol could be used. Thus, we could also write

$$\mathbf{b} = b_j\mathbf{e}_j. \tag{1.14}$$

The *scalar* (or *dot*) product of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = (a_i\mathbf{e}_i) \cdot (b_j\mathbf{e}_j) = a_i b_j(\mathbf{e}_i \cdot \mathbf{e}_j) = a_i b_j \delta_{ij} = a_i b_i, \tag{1.15}$$

or, in expanded form,

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3. \tag{1.16}$$

We define a third vector $\mathbf{c}$ as

$$\mathbf{c} = \mathbf{a} - \mathbf{b}, \tag{1.17}$$

as shown in Fig. 1, in which case

$$(a - b\cos\theta)^2 + (b\sin\theta)^2 = c^2, \tag{1.18}$$

where a, b, and c are the lengths of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, respectively. The *length* (or *magnitude*) of the vector $\mathbf{a}$ is

$$a = |\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_i a_i} = \sqrt{a_1^2 + a_2^2 + a_3^2}. \tag{1.19}$$

Eq. 1.18 can be expanded to yield

$$a^2 - 2ab\cos\theta + b^2\cos^2\theta + b^2\sin^2\theta = c^2 \tag{1.20}$$

or

$$c^2 = a^2 + b^2 - 2ab\cos\theta, \tag{1.21}$$

which is the *law of cosines*. We can further expand the law of cosines in terms of components to obtain

$$(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2ab\cos\theta \tag{1.22}$$

or

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = ab\cos\theta. \tag{1.23}$$

Thus, the dot product of two vectors can alternatively be expressed as

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = ab\cos\theta, \tag{1.24}$$

where θ is the angle between the two vectors.

The *vector* (or *cross*) product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}\times\mathbf{b} = a_i\mathbf{e}_i \times b_j\mathbf{e}_j = a_i b_j \mathbf{e}_i \times \mathbf{e}_j = a_i b_j e_{ijk}\mathbf{e}_k, \tag{1.25}$$

where the last result follows from Eq. 1.10. This result is expanded by summing on i, j, k to obtain

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{e}_1 + (a_3b_1 - a_1b_3)\mathbf{e}_2 + (a_1b_2 - a_2b_1)\mathbf{e}_3 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{1.26}$$

It is also shown in elementary vector analysis that

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|(\sin\theta)\mathbf{e}_n, \tag{1.27}$$

where $\mathbf{e}_n$ is the unit vector which is perpendicular to the plane formed by $\mathbf{a}$ and $\mathbf{b}$ and lies in the direction indicated by the right-hand rule if $\mathbf{a}$ is rotated into $\mathbf{b}$.

For three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, consider the triple scalar product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, where $\mathbf{c}$ is not the same vector defined in Eq. 1.17. This product represents the volume of the parallelepiped having $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ as edges (Fig. 2), since

$$|\mathbf{b}\times\mathbf{c}| = |\mathbf{b}||\mathbf{c}|\sin\theta \tag{1.28}$$

is the area of the parallelogram with sides $\mathbf{b}$ and $\mathbf{c}$, and $|\mathbf{a}|\cos\alpha$ is the height of the parallelepiped. Thus,

$$|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = |\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin\theta\cos\alpha = \text{height}\times\text{area} = \text{volume}. \tag{1.29}$$

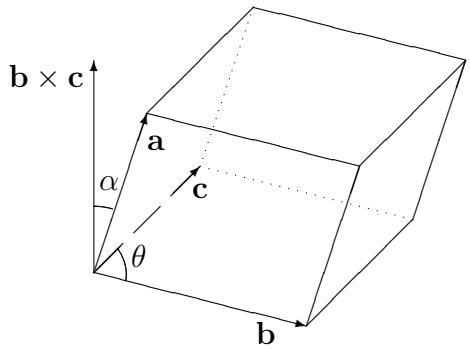


Figure 2: Parallelepiped and Triple Scalar Product.

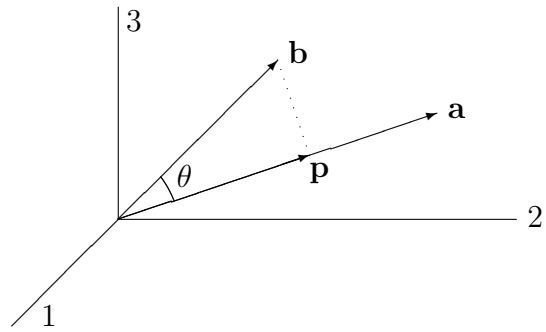


Figure 3: Projection Onto Line.

From Eq. 1.26, we obtain

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{1.30}$$

The volume property of the triple scalar product implies that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \tag{1.31}$$

Thus, in the triple scalar product, the dot and the cross can be interchanged without affecting the result.

Consider two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, as shown in Fig. 3. Let $\mathbf{p}$ be the vector obtained by projecting $\mathbf{b}$ onto $\mathbf{a}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $|\mathbf{b}| \cos\theta$, where θ is the angle between $\mathbf{b}$ and $\mathbf{a}$. Thus, the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\mathbf{p} = (|\mathbf{b}| \cos\theta) \frac{\mathbf{a}}{|\mathbf{a}|}, \tag{1.32}$$

where the fraction in this expression is the unit vector in the direction of $\mathbf{a}$. Since

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta, \tag{1.33}$$

Eq. 1.32 becomes

$$\mathbf{p} = \left(\mathbf{b} \cdot \frac{\mathbf{a}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|}. \tag{1.34}$$

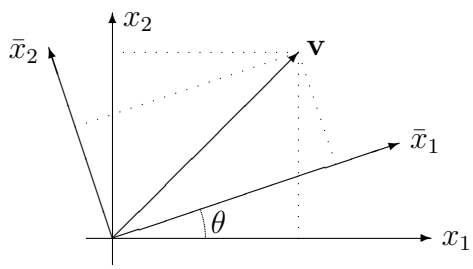


Figure 4: Change of Basis.

1.2 Change of Basis

On many occasions in engineering applications, including elasticity, the need arises to transform vectors and matrices from one coordinate system to another. Consider the vector $\mathbf{v}$ given by

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_i\mathbf{e}_i, \tag{1.35}$$

where $\mathbf{e}_i$ are the basis vectors, v_i are the components of $\mathbf{v}$, and the summation convention was used.

Since bases are not unique, we can express $\mathbf{v}$ in two different orthonormal bases:

$$\mathbf{v} = \sum_{i=1}^{3} v_i\mathbf{e}_i = \sum_{i=1}^{3} \bar{v}_i\bar{\mathbf{e}}_i, \tag{1.36}$$

where v_i are the components of $\mathbf{v}$ in the unbarred coordinate system, and $\bar{v}_i$ are the components in the barred system (Fig. 4). If we take the dot product of both sides of Eq. 1.36 with $\mathbf{e}_j$, we obtain

$$\sum_{i=1}^{3} v_i\,\mathbf{e}_i \cdot \mathbf{e}_j = \sum_{i=1}^{3} \bar{v}_i\,\bar{\mathbf{e}}_i \cdot \mathbf{e}_j, \tag{1.37}$$

where $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, and we define the 3×3 matrix $\mathbf{R}$ as

$$R_{ij} = \bar{\mathbf{e}}_i \cdot \mathbf{e}_j. \tag{1.38}$$

Thus, from Eq. 1.37,

$$v_j = \sum_{i=1}^{3} R_{ij}\bar{v}_i = \sum_{i=1}^{3} R^T_{ji}\bar{v}_i. \tag{1.39}$$

Since the matrix product

$$\mathbf{C} = \mathbf{AB} \tag{1.40}$$

can be written using subscript notation as

$$C_{ij} = \sum_{k=1}^{3} A_{ik}B_{kj}, \tag{1.41}$$

Eq. 1.39 is equivalent to the matrix product

$$\mathbf{v} = \mathbf{R}^T\bar{\mathbf{v}}. \tag{1.42}$$

Similarly, if we take the dot product of Eq. 1.36 with $\bar{\mathbf{e}}_j$, we obtain

$$\sum_{i=1}^{3} v_i\, \mathbf{e}_i \cdot \bar{\mathbf{e}}_j = \sum_{i=1}^{3} \bar{v}_i\, \bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j, \tag{1.43}$$

where $\bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j = \delta_{ij}$, and $\mathbf{e}_i \cdot \bar{\mathbf{e}}_j = R_{ji}$. Thus,

$$\bar{v}_j = \sum_{i=1}^{3} R_{ji} v_i \quad \text{or} \quad \bar{\mathbf{v}} = \mathbf{R}\mathbf{v} \quad \text{or} \quad \mathbf{v} = \mathbf{R}^{-1}\bar{\mathbf{v}}. \tag{1.44}$$

A comparison of Eqs. 1.42 and 1.44 yields

$$\mathbf{R}^{-1} = \mathbf{R}^T \quad \text{or} \quad \mathbf{R}\mathbf{R}^T = \mathbf{I} \quad \text{or} \quad \sum_{k=1}^{3} R_{ik} R_{jk} = \delta_{ij}, \tag{1.45}$$

where $\mathbf{I}$ is the identity matrix ($I_{ij} = \delta_{ij}$):

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.46}$$

This type of transformation is called an *orthogonal coordinate transformation* (OCT). A matrix $\mathbf{R}$ satisfying Eq. 1.45 is said to be an *orthogonal* matrix. That is, an orthogonal matrix is one whose inverse is equal to the transpose. $\mathbf{R}$ is sometimes called a *rotation matrix*.

For example, for the coordinate rotation shown in Fig. 4, in 3-D,

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.47}$$

In 2-D,

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{1.48}$$

and

$$\begin{cases} v_x = \bar{v}_x \cos\theta - \bar{v}_y \sin\theta \\ v_y = \bar{v}_x \sin\theta + \bar{v}_y \cos\theta. \end{cases} \tag{1.49}$$

We recall that the determinant of a matrix product is equal to the product of the determinants. Also, the determinant of the transpose of a matrix is equal to the determinant of the matrix itself. Thus, from Eq. 1.45,

$$\det(\mathbf{R}\mathbf{R}^T) = (\det \mathbf{R})(\det \mathbf{R}^T) = (\det \mathbf{R})^2 = \det \mathbf{I} = 1, \tag{1.50}$$

and we conclude that, for an orthogonal matrix $\mathbf{R}$,

$$\det \mathbf{R} = \pm 1. \tag{1.51}$$

The plus sign occurs for rotations, and the minus sign occurs for combinations of rotations and reflections that result in a net reflection. For example, the orthogonal matrix

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{1.52}$$

indicates a reflection in the z direction (i.e., the sign of the z component is changed).

Another property of orthogonal matrices that can be deduced directly from the definition, Eq. 1.45, is that the rows and columns of an orthogonal matrix must be unit vectors and mutually orthogonal. That is, the rows and columns form an orthonormal set.

We note that the length of a vector is unchanged under an orthogonal coordinate transformation, since the square of the length is given by

$$\bar{v}_i \bar{v}_i = R_{ij} v_j R_{ik} v_k = \delta_{jk} v_j v_k = v_j v_j, \tag{1.53}$$

where the summation convention was used. That is, the square of the length of a vector is the same in both coordinate systems.

To summarize, under an orthogonal coordinate transformation, vectors transform according to the rule

$$\bar{\mathbf{v}} = \mathbf{R}\mathbf{v} \quad \text{or} \quad \bar{v}_i = \sum_{j=1}^{3} R_{ij} v_j, \tag{1.54}$$

where

$$R_{ij} = \bar{\mathbf{e}}_i \cdot \mathbf{e}_j, \tag{1.55}$$

and

$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}. \tag{1.56}$$

A vector which transforms under an orthogonal coordinate transformation according to the rule $\bar{\mathbf{v}} = \mathbf{R}\mathbf{v}$ is defined as a tensor of rank 1. Examples include displacement vectors, velocity vectors, and force vectors. A tensor of rank 0 is a scalar (a quantity which is unchanged by an orthogonal coordinate transformation). For example, temperature and pressure are scalars, since $\bar{T} = T$ and $\bar{p} = p$.

We now introduce tensors of rank 2. Consider a matrix $\mathbf{M} = (M_{ij})$ which relates two vectors $\mathbf{u}$ and $\mathbf{v}$ by

$$\mathbf{v} = \mathbf{M}\mathbf{u} \quad \text{or} \quad v_i = \sum_{j=1}^{3} M_{ij} u_j \tag{1.57}$$

(i.e., the result of multiplying a matrix and a vector is a vector). Also, in a rotated coordinate system,

$$\bar{\mathbf{v}} = \bar{\mathbf{M}}\bar{\mathbf{u}}. \tag{1.58}$$

Since both $\mathbf{u}$ and $\mathbf{v}$ are vectors (tensors of rank 1), Eq. 1.57 implies

$$\mathbf{R}^T\bar{\mathbf{v}} = \mathbf{M}\mathbf{R}^T\bar{\mathbf{u}} \quad \text{or} \quad \bar{\mathbf{v}} = \mathbf{R}\mathbf{M}\mathbf{R}^T\bar{\mathbf{u}}. \tag{1.59}$$

By comparing Eqs. 1.58 and 1.59, we conclude that

$$\bar{\mathbf{M}} = \mathbf{R}\mathbf{M}\mathbf{R}^T \tag{1.60}$$

or, in index notation,

$$\bar{M}_{ij} = \sum_{k=1}^{3}\sum_{l=1}^{3} R_{ik}R_{jl}M_{kl}, \tag{1.61}$$

which is the transformation rule for a tensor of rank 2. In general, a tensor of rank n, which has n indices, transforms under an orthogonal coordinate transformation according to the rule

$$\bar{A}_{ij\cdots k} = \sum_{p=1}^{3}\sum_{q=1}^{3}\cdots\sum_{r=1}^{3} R_{ip}R_{jq}\cdots R_{kr}A_{pq\cdots r}. \tag{1.62}$$

Notice that the subscripts of $\bar{A}$ appear as the first subscripts of each $\mathbf{R}$ matrix, and the subscripts of A appear as the second subscripts of each $\mathbf{R}$ matrix.

The *trace* of a matrix is defined as the sum of the diagonal terms. In index notation,

$$\operatorname{tr}\bar{\mathbf{M}} = \bar{M}_{ii} = R_{ik}R_{il}M_{kl} = \delta_{kl}M_{kl} = M_{kk} = \operatorname{tr}\mathbf{M}, \tag{1.63}$$

which is a scalar. Thus, the trace of a matrix (a tensor of rank 2) is invariant under an orthogonal coordinate transformation. The trace is one of three invariants associated with 3×3 matrices.

An *isotropic tensor* is a tensor which is independent of coordinate system (i.e., invariant under an orthogonal coordinate transformation). For example, the Kronecker delta δ_{ij} is an isotropic tensor, since $\bar{\delta}_{ij} = \delta_{ij}$, and

$$\bar{\delta}_{ij} = R_{ik}R_{jl}\delta_{kl} = R_{ik}R_{jk} = \delta_{ij}. \tag{1.64}$$

Hence, δ_{ij} is a second rank tensor and isotropic. In matrix notation,

$$\bar{\mathbf{I}} = \mathbf{R}\mathbf{I}\mathbf{R}^T = \mathbf{R}\mathbf{R}^T = \mathbf{I}. \tag{1.65}$$

1.3 Symmetry and Skew-Symmetry

A matrix $\mathbf{S}$ is defined as *symmetric* if $\mathbf{S} = \mathbf{S}^T$, i.e., $S_{ij} = S_{ji}$. A matrix $\mathbf{A}$ is defined as *skew-symmetric* (or *antisymmetric*) if $\mathbf{A} = -\mathbf{A}^T$, i.e., $A_{ij} = -A_{ji}$. Note that a skew-symmetric matrix necessarily has the form

$$\mathbf{A} = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix}. \tag{1.66}$$

Any matrix $\mathbf{M}$ can be written as the unique sum of symmetric and skew-symmetric matrices

$$\mathbf{M} = \mathbf{S} + \mathbf{A}, \tag{1.67}$$

where

$$\mathbf{S} = (\mathbf{M} + \mathbf{M}^T)/2 = \mathbf{S}^T, \tag{1.68}$$

$$\mathbf{A} = (\mathbf{M} - \mathbf{M}^T)/2 = -\mathbf{A}^T. \tag{1.69}$$

For example, for a 3×3 matrix $\mathbf{M}$,

$$\mathbf{M} = \begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 8 \\ 9 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 8 \\ 3 & 2 & 7 \\ 8 & 7 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \mathbf{S} + \mathbf{A}. \tag{1.70}$$

Note that, if $\mathbf{A}$ is skew-symmetric, $\mathbf{x} \cdot \mathbf{A}\mathbf{x} = 0$ for all $\mathbf{x}$. To prove this assertion, we write

$$\mathbf{x} \cdot \mathbf{A}\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x} = \left(\mathbf{x}^T\mathbf{A}\mathbf{x}\right)^T = \mathbf{x}^T\mathbf{A}^T\mathbf{x} = -\mathbf{x}^T\mathbf{A}\mathbf{x}, \tag{1.71}$$

where $\mathbf{x}^T\mathbf{A}\mathbf{x}$ is a scalar equal to its own transpose. Since this quantity is also equal to its own negative, it must vanish, and the assertion is proved. Thus, for a general matrix $\mathbf{M} = \mathbf{S} + \mathbf{A}$,

$$\mathbf{x} \cdot \mathbf{M}\mathbf{x} = \mathbf{x} \cdot \mathbf{S}\mathbf{x}, \tag{1.72}$$

where $\mathbf{S}$ is the symmetric part of $\mathbf{M}$. The matrix product $\mathbf{x} \cdot \mathbf{M}\mathbf{x} = x_i M_{ij} x_j$ is referred to as a *quadratic form*. In expanded form, the quadratic form is

$$\begin{aligned} \mathbf{x} \cdot \mathbf{M}\mathbf{x} = {} & M_{11}x_1^2 + M_{22}x_2^2 + M_{33}x_3^2 \\ & + (M_{12} + M_{21})x_1x_2 + (M_{23} + M_{32})x_2x_3 + (M_{13} + M_{31})x_1x_3. \end{aligned} \tag{1.73}$$

Note that the quadratic form can be written in three different notations:

vector (dyadic) notation	$\mathbf{x} \cdot \mathbf{M}\mathbf{x}$
matrix notation	$\mathbf{x}^T\mathbf{M}\mathbf{x}$
index notation	$x_i M_{ij} x_j$ or $M_{ij} x_i x_j$

A matrix $\mathbf{M}$ is defined as *positive definite* if $\mathbf{x} \cdot \mathbf{M}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. $\mathbf{M}$ is *positive semi-definite* if $\mathbf{x} \cdot \mathbf{M}\mathbf{x} \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$.

1.4 Derivatives and Divergence

Consider the scalar function $\phi(x_1, x_2, x_3)$. According to the chain rule,

$$\frac{\partial \phi}{\partial \bar{x}_i} = \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial \bar{x}_i}, \tag{1.74}$$

where $x_j = R_{kj}\bar{x}_k$. Hence,

$$\frac{\partial x_j}{\partial \bar{x}_i} = R_{kj}\frac{\partial \bar{x}_k}{\partial \bar{x}_i} = R_{kj}\delta_{ki} = R_{ij}. \tag{1.75}$$

Thus, from Eq. 1.74,

$$\frac{\partial \phi}{\partial \bar{x}_i} = R_{ij}\frac{\partial \phi}{\partial x_j}, \tag{1.76}$$

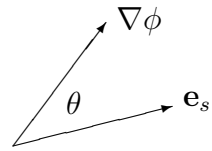


Figure 5: The Directional Derivative.

from which we conclude that the partial derivative $\partial\phi/\partial x_i$ is a tensor of rank 1.

The vector operator *del* is defined in Cartesian coordinates as

$$\nabla = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3} = \mathbf{e}_i\frac{\partial}{\partial x_i}. \tag{1.77}$$

The *gradient* of a scalar function $\phi(x_1, x_2, x_3)$ is defined as $\nabla\phi$, so that, in Cartesian coordinates,

$$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 = \mathbf{e}_i\frac{\partial\phi}{\partial x_i}. \tag{1.78}$$

This vector has as its components the rates of change of ϕ with respect to distance in the x_1, x_2, and x_3 directions, respectively. Note that, in a rotated Cartesian coordinate system,

$$\nabla\phi = \bar{\mathbf{e}}_i\frac{\partial\phi}{\partial\bar{x}_i}. \tag{1.79}$$

The *directional derivative* measures the rate of change of a scalar function, say $\phi(x_1, x_2, x_3)$, with respect to distance in any arbitrary direction s (Fig. 5) and is given by

$$\frac{\partial\phi}{\partial s} = \mathbf{e}_s \cdot \nabla\phi = |\mathbf{e}_s||\nabla\phi|\cos\theta = |\nabla\phi|\cos\theta, \tag{1.80}$$

where $\mathbf{e}_s$ is the unit vector in the s direction, and θ is the angle between the two vectors $\nabla\phi$ and $\mathbf{e}_s$. Thus, the maximum rate of change of ϕ is in the direction of $\nabla\phi$.

Given a vector function (field) $\mathbf{f}(x_1, x_2, x_3)$, the *divergence* of $\mathbf{f}$ is defined as $\nabla\cdot\mathbf{f}$, so that, in Cartesian coordinates,

$$\text{div } \mathbf{f} = \nabla\cdot\mathbf{f} = \left(\mathbf{e}_i\frac{\partial}{\partial x_i}\right)\cdot(f_j\mathbf{e}_j) = \frac{\partial f_j}{\partial x_i}\delta_{ij} = \frac{\partial f_i}{\partial x_i} \tag{1.81}$$

or, in expanded form,

$$\nabla\cdot\mathbf{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}. \tag{1.82}$$

We denote the partial derivative with respect to the ith Cartesian coordinate direction with the comma notation

$$\frac{\partial\phi}{\partial x_i} = \phi_{,i}. \tag{1.83}$$

Using this notation, the divergence becomes (in Cartesian coordinates)

$$\nabla\cdot\mathbf{f} = f_{i,i}. \tag{1.84}$$

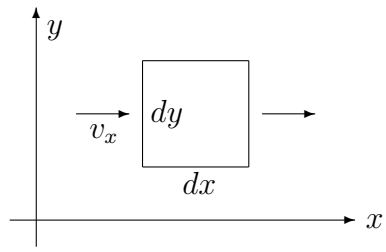


Figure 6: Fluid Flow Through Small Cube.

The *Laplacian* of the scalar field $\phi(x_1, x_2, x_3)$, denoted $\nabla^2\phi$, is defined as

$$\nabla^2\phi = \nabla \cdot \nabla\phi = \left(\mathbf{e}_i \frac{\partial}{\partial x_i}\right) \cdot \left(\mathbf{e}_j \frac{\partial \phi}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\right)\delta_{ij} = \phi_{,ii} \tag{1.85}$$

or, in expanded form,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}. \tag{1.86}$$

1.5 The Divergence Theorem

Consider the steady (time-independent) motion of a fluid of density $\rho(x, y, z)$ and velocity

$$\mathbf{v} = v_x(x, y, z)\mathbf{e}_x + v_y(x, y, z)\mathbf{e}_y + v_z(x, y, z)\mathbf{e}_z, \tag{1.87}$$

where, for this section, it is convenient to denote the Cartesian coordinates as (x, y, z) rather than (x_1, x_2, x_3). In the small cube of dimensions dx, dy, dz (Fig. 6)[10], the mass entering the face $dy\,dz$ on the left per unit time is $\rho v_x\,dy\,dz$. The mass exiting per unit time on the right is given by the two-term Taylor series expansion as

$$\left[\rho v_x + \frac{\partial(\rho v_x)}{\partial x}\,dx\right] dy\,dz,$$

so that the loss of mass per unit time in the x-direction is

$$\frac{\partial}{\partial x}(\rho v_x)\,dx\,dy\,dz.$$

If we also take into consideration the other two faces, the total loss of mass per unit time for the small cube is

$$\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right] dx\,dy\,dz = \nabla \cdot (\rho\mathbf{v})\,dx\,dy\,dz = \nabla \cdot (\rho\mathbf{v})\,dV,$$

where V is volume.

If we now consider a volume V bounded by a closed surface S (Fig. 7), the total loss of mass per unit time is

$$\int_V \nabla \cdot (\rho\mathbf{v})\,dV.$$

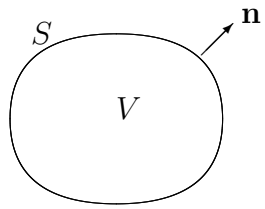


Figure 7: A Closed Volume Bounded by a Surface S.

However, the loss of fluid in V must be due to the flow of fluid through the boundary S. The outward flow of mass per unit time through a differential element of surface area dS is $\rho\mathbf{v}\cdot\mathbf{n}\,dS$, so that the total loss of mass per unit time through the boundary is

$$\oint_S \rho\mathbf{v}\cdot\mathbf{n}\,dS,$$

where $\mathbf{n}$ is the unit outward normal at a point on S, $\mathbf{v}\cdot\mathbf{n}$ is the normal component of $\mathbf{v}$, and the circle in the integral sign indicates that the surface integral is for a closed surface. Equating the last two expressions, we have

$$\int_V \nabla\cdot(\rho\mathbf{v})\,dV = \oint_S \rho\mathbf{v}\cdot\mathbf{n}\,dS, \tag{1.88}$$

or, in general, for any vector field $\mathbf{f}(x,y,z)$ defined in a volume V bounded by a closed surface S,

$$\int_V \nabla\cdot\mathbf{f}\,dV = \oint_S \mathbf{f}\cdot\mathbf{n}\,dS. \tag{1.89}$$

In index notation, this last equation is

$$\int_V f_{i,i}\,dV = \oint_S f_i n_i\,dS. \tag{1.90}$$

This is the *divergence theorem*, one of the fundamental integral theorems of mathematical physics. The divergence theorem is also known as the Green-Gauss theorem.

Related theorems are the *gradient theorem*, which states that

$$\int_V \nabla f\,dV = \oint_S f\mathbf{n}\,dS, \tag{1.91}$$

for f a scalar function, and the *curl theorem*:

$$\int_V \nabla\times\mathbf{f}\,dV = \oint_S \mathbf{n}\times\mathbf{f}\,dS. \tag{1.92}$$

To aid in the interpretation of the divergence of a vector field, consider a small volume ΔV surrounding a point P. From the divergence theorem,

$$(\nabla\cdot\mathbf{f})_P \approx \frac{1}{\Delta V}\oint_S \mathbf{f}\cdot\mathbf{n}\,dS, \tag{1.93}$$

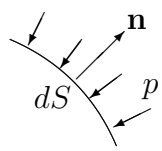


Figure 8: Pressure on Differential Element of Surface.

which implies that the divergence of $\mathbf{f}$ can be interpreted as the net outward flow (flux) of $\mathbf{f}$ at P per unit volume. (If $\mathbf{f}$ is the velocity field $\mathbf{v}$, the right-hand side integrand is the normal component v_n of velocity.) Similarly, we can see the plausibility of the divergence theorem by observing that the divergence multiplied by the volume element for each elemental cell is the net surface integral out of that cell. When summed by integration, all internal contributions cancel, since the flow out of one cell goes into another, and only the external surface contributions remain.

To illustrate the use of the integral theorems, consider a body of arbitrary shape to which the pressure $p(x, y, z)$ is applied. In general, p can be position-dependent. The force on a differential element of surface dS (Fig. 8) is $p\,dS$. Since this force acts in the $-\mathbf{n}$ direction, the resultant force $\mathbf{F}$ acting on the body is then

$$\mathbf{F} = -\oint_S p\mathbf{n}\,dS = -\int_V \nabla p\,dV, \tag{1.94}$$

where the second equation follows from the gradient theorem. This relation is quite general, since it applies to arbitrary pressure distributions. For the special case of uniform pressure ($p =$ constant), $\mathbf{F} = \mathbf{0}$; i.e., the resultant force of a uniform pressure load on an arbitrary body is zero.

1.6 Eigenvalue Problems

If, for a square matrix $\mathbf{A}$ of order n, there exists a vector $\mathbf{x} \neq \mathbf{0}$ and a number λ such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \tag{1.95}$$

λ is called an eigenvalue of $\mathbf{A}$, and $\mathbf{x}$ is the corresponding eigenvector. Note that Eq. 1.95 can alternatively be written in the form

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{I}\mathbf{x}, \tag{1.96}$$

where $\mathbf{I}$ is the identity matrix, or

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}, \tag{1.97}$$

which is a system of n linear algebraic equations. If the coefficient matrix $\mathbf{A} - \lambda\mathbf{I}$ were nonsingular, the unique solution of Eq. 1.97 would be $\mathbf{x} = \mathbf{0}$, which is not of interest. Thus, to obtain a nonzero solution of the eigenvalue problem, $\mathbf{A} - \lambda\mathbf{I}$ must be singular, which

implies that

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0. \tag{1.98}$$

This equation is referred to as the *characteristic equation* or *characteristic polynomial* for the eigenvalue problem. We note that this determinant, when expanded, yields a polynomial of degree n. The eigenvalues are thus the roots of this polynomial. Since a polynomial of degree n has n roots, $\mathbf{A}$ has n eigenvalues (not necessarily real).

Eigenvalue problems arise in many physical applications, including free vibrations of mechanical systems, buckling of structures, and the calculation of principal axes of stress, strain, and inertia.

To illustrate the algebraic calculation of eigenvalues and eigenvectors, consider the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \tag{1.99}$$

from which it follows that

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)-1 = \lambda^2-3\lambda+1 = 0. \tag{1.100}$$

Thus,

$$\lambda = \frac{3 \pm \sqrt{5}}{2} \approx 0.382 \text{ and } 2.618\,. \tag{1.101}$$

To obtain the eigenvectors, we solve Eq. 1.97 for each eigenvalue found. For the first eigenvalue, $\lambda = 0.382$, we obtain

$$\begin{bmatrix} 1.618 & -1 \\ -1 & 0.618 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{1.102}$$

which is a redundant system of equations satisfied by

$$\mathbf{x}^{(1)} = \begin{Bmatrix} 0.618 \\ 1 \end{Bmatrix}. \tag{1.103}$$

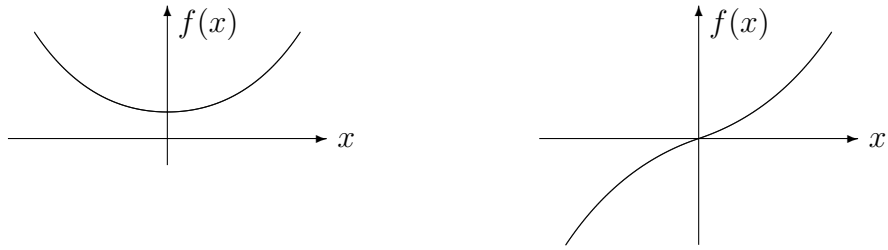
The superscript indicates that this is the eigenvector associated with the first eigenvalue. Note that, since the eigenvalue problem is a homogeneous problem, any multiple of $\mathbf{x}^{(1)}$ is also an eigenvector. For the second eigenvalue, $\lambda = 2.618$, we obtain

$$\begin{bmatrix} -0.618 & -1 \\ -1 & -1.618 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{1.104}$$

a solution of which is

$$\mathbf{x}^{(2)} = \begin{Bmatrix} 1 \\ -0.618 \end{Bmatrix}. \tag{1.105}$$

The determinant approach used above is fine for small matrices but not well-suited to numerical computation involving large matrices.



(a) Even (b) Odd

Figure 9: Examples of Even and Odd Functions.

1.7 Even and Odd Functions

A function $f(x)$ is *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$, as illustrated in Fig. 9. Geometrically, an even function is *symmetrical* with respect to the line $x = 0$, and an odd function is *antisymmetrical*. For example, $\cos x$ is even, and $\sin x$ is odd.

Even and odd functions have the following useful properties:

1. If f is even and *smooth* (continuous first derivative), $f'(0) = 0$.
2. If f is odd, $f(0) = 0$.
3. The product of two even functions is even.
4. The product of two odd functions is even.
5. The product of an even and an odd function is odd.
6. The derivative of an even function is odd.
7. The derivative of an odd function is even.
8. If f is even,
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx. \tag{1.106}$$
9. If f is odd,
$$\int_{-a}^{a} f(x)\,dx = 0. \tag{1.107}$$

Most functions are neither even nor odd. However, any function $f(x)$ can be represented as the unique sum of even and odd functions

$$f(x) = f_e(x) + f_o(x), \tag{1.108}$$

where $f_e(x)$ is the even part of f given by

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)], \tag{1.109}$$

and $f_o(x)$ is the odd part of f given by

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)]. \tag{1.110}$$

2 Strain

2.1 Admissible Deformations

Consider an object which undergoes a deformation. Let (x_1, x_2, x_3) denote the coordinates of a point P in the undeformed state, and let (ξ_1, ξ_2, ξ_3) denote the coordinates of the same point after deformation. Thus,

$$\begin{cases} \xi_1 = \xi_1(x_1, x_2, x_3) \\ \xi_2 = \xi_2(x_1, x_2, x_3) \\ \xi_3 = \xi_3(x_1, x_2, x_3), \end{cases} \tag{2.1}$$

and, inversely,

$$\begin{cases} x_1 = x_1(\xi_1, \xi_2, \xi_3) \\ x_2 = x_2(\xi_1, \xi_2, \xi_3) \\ x_3 = x_3(\xi_1, \xi_2, \xi_3). \end{cases} \tag{2.2}$$

To compute the derivatives $\partial/\partial x_1$, $\partial/\partial x_2$, and $\partial/\partial x_3$, we use the chain rule in the form

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_1} + \frac{\partial}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_1} + \frac{\partial}{\partial \xi_3}\frac{\partial \xi_3}{\partial x_1}, \tag{2.3}$$

with similar relationships for $\partial/\partial x_2$ and $\partial/\partial x_3$. In matrix form, we have

$$\begin{Bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} & \dfrac{\partial \xi_3}{\partial x_1} \\ \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\partial \xi_3}{\partial x_2} \\ \dfrac{\partial \xi_1}{\partial x_3} & \dfrac{\partial \xi_2}{\partial x_3} & \dfrac{\partial \xi_3}{\partial x_3} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial \xi_1} \\ \dfrac{\partial}{\partial \xi_2} \\ \dfrac{\partial}{\partial \xi_3} \end{Bmatrix}, \tag{2.4}$$

where the 3×3 matrix is referred to as the *Jacobian matrix*:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} & \dfrac{\partial \xi_3}{\partial x_1} \\ \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\partial \xi_3}{\partial x_2} \\ \dfrac{\partial \xi_1}{\partial x_3} & \dfrac{\partial \xi_2}{\partial x_3} & \dfrac{\partial \xi_3}{\partial x_3} \end{bmatrix}. \tag{2.5}$$

We note that, to have a transformation for which the inverse exists, $\mathbf{J}^{-1}$ must exist, i.e.,

$$\det \mathbf{J} = |\mathbf{J}| \neq 0. \tag{2.6}$$

If the body is not deformed at all (i.e., $\xi_1 = x_1$, $\xi_2 = x_2$, and $\xi_3 = x_3$), $|\mathbf{J}| = 1$. Thus, since the deformation is a continuous function of time, and $|\mathbf{J}| = 1$ initially, we require $|\mathbf{J}| > 0$ for a physically realizable deformation.

The components of displacement of the point P are the differences between the old and new coordinates:

$$\begin{cases} u_1 = \xi_1 - x_1 \\ u_2 = \xi_2 - x_2 \\ u_3 = \xi_3 - x_3, \end{cases} \tag{2.7}$$

in which case

$$\frac{\partial \xi_1}{\partial x_1} = 1 + \frac{\partial u_1}{\partial x_1}, \quad \frac{\partial \xi_2}{\partial x_1} = \frac{\partial u_2}{\partial x_1} \tag{2.8}$$

and similarly for the other derivatives. Thus, we could alternatively write the condition for a physically possible deformation as

$$|\mathbf{J}| = \begin{vmatrix} 1 + \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_1}{\partial x_2} & 1 + \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_3}{\partial x_2} \\ \dfrac{\partial u_1}{\partial x_3} & \dfrac{\partial u_2}{\partial x_3} & 1 + \dfrac{\partial u_3}{\partial x_3} \end{vmatrix} > 0. \tag{2.9}$$

For example, consider the displacement field

$$\begin{cases} u_1 = x_1 - 2x_2 \\ u_2 = 3x_1 + 2x_2 \\ u_3 = 5x_3, \end{cases} \tag{2.10}$$

for which

$$|\mathbf{J}| = \begin{vmatrix} 2 & 3 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 6 \end{vmatrix} = 72 > 0. \tag{2.11}$$

Thus, this displacement field is admissible.

Geometrically, the determinant of the Jacobian in Eq. 2.5 can be seen to be the ratio of two volumes (the ratio of the new volume to the old volume). If (ξ_1, ξ_2, ξ_3) are curvilinear coordinates, the vectors

$$\mathbf{dx_1} = \begin{Bmatrix} \dfrac{\partial \xi_1}{\partial x_1} \\ \dfrac{\partial \xi_2}{\partial x_1} \\ \dfrac{\partial \xi_3}{\partial x_1} \end{Bmatrix} dx_1, \quad \mathbf{dx_2} = \begin{Bmatrix} \dfrac{\partial \xi_1}{\partial x_2} \\ \dfrac{\partial \xi_2}{\partial x_2} \\ \dfrac{\partial \xi_3}{\partial x_2} \end{Bmatrix} dx_2, \quad \mathbf{dx_3} = \begin{Bmatrix} \dfrac{\partial \xi_1}{\partial x_3} \\ \dfrac{\partial \xi_2}{\partial x_3} \\ \dfrac{\partial \xi_3}{\partial x_3} \end{Bmatrix} dx_3 \tag{2.12}$$

are tangent to the x_1, x_2, and x_3 coordinate curves, respectively. (The x_1 curve is the curve obtained by changing x_1 while x_2 and x_3 are held fixed.) For example, in two dimensions (Fig. 10), if x_1 is increased by Δx_1,

$$\frac{\Delta \xi_1}{\Delta x_1} = \cos\theta, \quad \frac{\Delta \xi_2}{\Delta x_1} = \sin\theta, \tag{2.13}$$

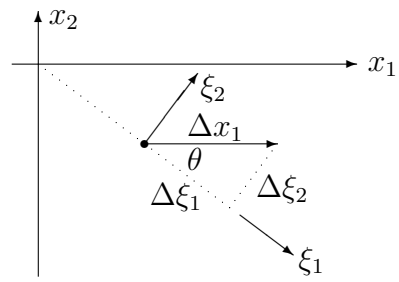


Figure 10: 2-D Coordinate Transformation.

which are the components of the tangent to the x_1 curve in the ξ_1-ξ_2 coordinates. From the discussion of the triple scalar product, Eq. 1.30, the element of volume is

$$dV = \mathbf{dx_1} \cdot (\mathbf{dx_2} \times \mathbf{dx_3}) = \begin{vmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_1} & \dfrac{\partial \xi_3}{\partial x_1} \\ \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\partial \xi_3}{\partial x_2} \\ \dfrac{\partial \xi_1}{\partial x_3} & \dfrac{\partial \xi_2}{\partial x_3} & \dfrac{\partial \xi_3}{\partial x_3} \end{vmatrix} dx_1\, dx_2\, dx_3 = |\mathbf{J}|\, dx_1\, dx_2\, dx_3. \tag{2.14}$$

Thus, the requirement that $|\mathbf{J}| > 0$ for an admissible deformation is equivalent to requiring positive volume.

Since $|\mathbf{J}|$ is a ratio of volumes, it is the ratio of new volume to old volume for a transformation. For a volume V that increases by ΔV,

$$|\mathbf{J}| = \frac{V + \Delta V}{V} = 1 + \frac{\Delta V}{V}, \tag{2.15}$$

where $\Delta V/V$ is referred to as the volumetric strain. If we expand the determinant in Eq. 2.9, we obtain

$$\begin{aligned} |\mathbf{J}| =& 1 + u_{i,i} + u_{1,1}u_{2,2} + u_{2,2}u_{3,3} + u_{3,3}u_{1,1} - u_{1,2}u_{2,1} - u_{2,3}u_{3,2} - u_{3,1}u_{1,3} + u_{1,1}u_{2,2}u_{3,3} \\ & - u_{1,1}u_{2,3}u_{3,2} - u_{1,2}u_{2,1}u_{3,3} + u_{1,2}u_{3,1}u_{2,3} + u_{1,3}u_{2,1}u_{3,2} - u_{1,3}u_{3,1}u_{2,2}. \end{aligned} \tag{2.16}$$

Thus, the volumetric strain is

$$\begin{aligned} \frac{\Delta V}{V} =& u_{i,i} + u_{1,1}u_{2,2} + u_{2,2}u_{3,3} + u_{3,3}u_{1,1} - u_{1,2}u_{2,1} - u_{2,3}u_{3,2} - u_{3,1}u_{1,3} + u_{1,1}u_{2,2}u_{3,3} \\ & - u_{1,1}u_{2,3}u_{3,2} - u_{1,2}u_{2,1}u_{3,3} + u_{1,2}u_{3,1}u_{2,3} + u_{1,3}u_{2,1}u_{3,2} - u_{1,3}u_{3,1}u_{2,2}. \end{aligned} \tag{2.17}$$

For small displacements, the product terms are small, and

$$\frac{\Delta V}{V} = u_{i,i}. \tag{2.18}$$

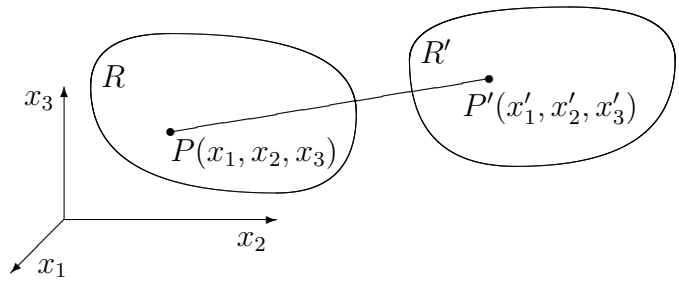


Figure 11: Geometry of Deformation.

2.2 Affine Transformations

Consider a body which undergoes a deformation. Let x_i $(i = 1, 2, 3)$ denote the set of original coordinates before deformation. Let $x'_i = x'_i(x_1, x_2, x_3)$ denote the new coordinates of the point (x_1, x_2, x_3) after deformation. Here we are concerned only with the geometry of the deformation. Neither the causes (e.g., forces or temperature changes) which give rise to deformation nor the law which governs the body's resistance to deformation are at issue now. We assume that $x'_i = x'_i(x_1, x_2, x_3)$ is smooth and single-valued. Thus, as shown in Fig. 11, the point P with coordinates (x_1, x_2, x_3) is transformed to the point P' with coordinates (x'_1, x'_2, x'_3).

A special case of this transformation occurs when $x'_i(x_1, x_2, x_3)$ is linear and is called an *affine transformation*:

$$x'_i = x_i + \alpha_{i0} + \alpha_{ij} x_j, \tag{2.19}$$

where x'_i is a new coordinate $(i = 1, 2, 3)$, x_i is an old coordinate, α_{i0} is a rigid body translation, and α_{ij} is a transformation matrix which includes the effects of both rigid body rotation and stretching. This equation could alternatively be written in the form

$$x'_i = \alpha_{i0} + (\delta_{ij} + \alpha_{ij}) x_j, \tag{2.20}$$

or, in matrix notation,

$$\mathbf{x}' = \boldsymbol{\alpha}_0 + (\mathbf{I} + \boldsymbol{\alpha})\mathbf{x}. \tag{2.21}$$

For example, in two dimensions, the matrix

$$\mathbf{I} + \boldsymbol{\alpha} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = c\mathbf{I} \tag{2.22}$$

represents a uniform stretching by the scalar factor c. The matrix

$$\mathbf{I} + \boldsymbol{\alpha} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.23}$$

represents a 90° counter-clockwise rotation, since

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} -x_2 \\ x_1 \end{Bmatrix}. \tag{2.24}$$

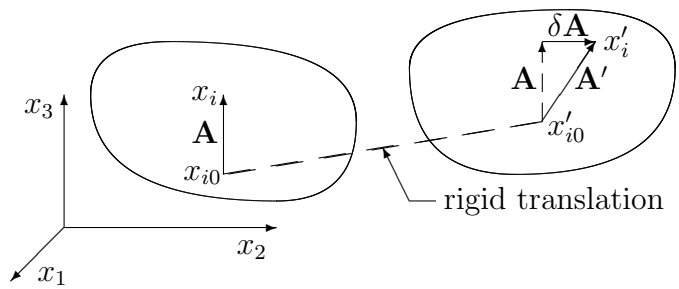


Figure 12: Affine Transformation.

We require the deformation, Eq. 2.20, to be reversible; i.e., we must be able to solve for x_i in terms of x_i':

$$x_i = \beta_{i0} + (\delta_{ij} + \beta_{ij})x_j', \tag{2.25}$$

which is also a linear transformation.

Two properties of affine transformations are of interest:

1. Planes transform into planes. We prove this assertion by substituting Eq. 2.25 into the general equation for a plane
$$Ax + By + Cz = D \tag{2.26}$$
and observing that another linear equation results.

2. Straight lines transform into straight lines. This statement is a consequence of the first property, since lines are intersections of planes.

Thus a vector $\mathbf{A} = A_i\mathbf{e}_i$ transforms into another vector $\mathbf{A}' = A_i'\mathbf{e}_i$.

Consider a body which undergoes a motion consisting of a translation, rotation, and deformation (Fig. 12). Let $\mathbf{A}$ be the vector in the undeformed body from the point with coordinates x_{i0} to the point with coordinates x_i:

$$A_i = x_i - x_{i0}. \tag{2.27}$$

Similarly, let $\mathbf{A}'$ be the vector in the translated and deformed body from the point with coordinates x_{i0}' to the point with coordinates x_i':

$$\begin{aligned} A_i' &= x_i' - x_{i0}' && (2.28)\\ &= (x_i + \alpha_{i0} + \alpha_{ij}x_j) - (x_{i0} + \alpha_{i0} + \alpha_{ij}x_{j0}) && (2.29)\\ &= (x_i - x_{i0}) + \alpha_{ij}(x_j - x_{j0}) && (2.30)\\ &= A_i + \alpha_{ij}A_j. && (2.31) \end{aligned}$$

Since the rigid body translation component has cancelled, the components of the rotation/stretch vector are then

$$\delta A_i = A_i' - A_i = \alpha_{ij}A_j \quad (i = 1, 2, 3). \tag{2.32}$$

We now want to separate the rotation from the stretch in the α_{ij}. Let A denote the length of the vector $\mathbf{A}$, i.e., $A = |\mathbf{A}|$. Thus,

$$A^2 = \mathbf{A} \cdot \mathbf{A} = A_iA_i, \tag{2.33}$$

from which it follows (by differentiation) that

$$2A\,\delta A = \delta A_i\,A_i + A_i\,\delta A_i = 2A_i\,\delta A_i, \tag{2.34}$$

where δA is the change in length of $\mathbf{A}$. Hence, from Eq. 2.32,

$$A\,\delta A = A_i\,\delta A_i = \alpha_{ij}A_iA_j. \tag{2.35}$$

For rigid motion (rotation without stretch), $\delta A = 0$, and

$$\alpha_{ij}A_iA_j = 0 \text{ for all } A_i. \tag{2.36}$$

That is, for rigid motion,

$$\alpha_{11}A_1^2 + \alpha_{22}A_2^2 + \alpha_{33}A_3^2 + (\alpha_{12}+\alpha_{21})A_1A_2 + (\alpha_{13}+\alpha_{31})A_1A_3 + (\alpha_{23}+\alpha_{32})A_2A_3 = 0 \tag{2.37}$$

for all A_i, which implies that each coefficient must vanish:

$$\alpha_{11} = \alpha_{22} = \alpha_{33} = 0, \quad \alpha_{12} = -\alpha_{21}, \quad \alpha_{13} = -\alpha_{31}, \quad \alpha_{23} = -\alpha_{32} \tag{2.38}$$

or $\alpha_{ij} = -\alpha_{ji}$ (skew-symmetric). Since any matrix can be decomposed uniquely into the sum of symmetric and skew-symmetric matrices, we write

$$\alpha_{ij} = \frac{1}{2}(\alpha_{ij}+\alpha_{ji}) + \frac{1}{2}(\alpha_{ij}-\alpha_{ji}) = \varepsilon_{ij} + \omega_{ij}, \tag{2.39}$$

where the first term is symmetric and represents the deformation, and the second term is skew-symmetric and represents the rotation. Thus we have defined

$$\varepsilon_{ij} = \frac{1}{2}(\alpha_{ij}+\alpha_{ji}) = \varepsilon_{ji}, \tag{2.40}$$

$$\omega_{ij} = \frac{1}{2}(\alpha_{ij}-\alpha_{ji}) = -\omega_{ji}, \tag{2.41}$$

where ε_{ij} are called the *components of the strain tensor*, although we have not proved yet that this is a tensor.

2.3 Geometrical Interpretations of Strain Components

From Eqs. 2.35 and 2.39,

$$A\,\delta A = \alpha_{ij}A_iA_j = (\varepsilon_{ij}+\omega_{ij})A_iA_j = \varepsilon_{ij}A_iA_j + \omega_{ij}A_iA_j, \tag{2.42}$$

where the last quadratic form is zero, since ω_{ij} is skew-symmetric. Hence,

$$A\,\delta A = \varepsilon_{ij}A_iA_j. \tag{2.43}$$

If we divide this equation by A^2, we obtain the change in length per unit length

$$\frac{\delta A}{A} = \frac{\varepsilon_{ij}A_iA_j}{A^2}. \tag{2.44}$$

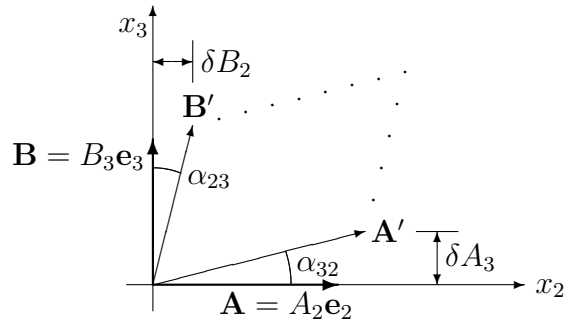


Figure 13: Geometrical Interpretation of Shear Strain.

For example, let $A_1 \neq 0$ and $A_2 = A_3 = 0$, in which case $A = A_1$ and

$$\frac{\delta A}{A} = \varepsilon_{11}. \tag{2.45}$$

That is, the strain ε_{11} represents the change in length per unit length in the x_1 direction. Similarly, the strains ε_{22} and ε_{33} represent the change in length per unit length in the x_2 and x_3 directions, respectively.

To interpret the off-diagonal component of strain ε_{23}, for example, consider the two vectors

$$\mathbf{A} = A_2\mathbf{e}_2, \quad \mathbf{B} = B_3\mathbf{e}_3, \tag{2.46}$$

where $A_1 = A_3 = B_1 = B_2 = 0$. From Eq. 2.32 ($\delta A_i = \alpha_{ij}A_j$),

$$\delta A_3 = \alpha_{32}A_2, \quad \delta B_2 = \alpha_{23}B_3, \tag{2.47}$$

which imply the two vectors $\mathbf{A}$ and $\mathbf{B}$ are rotated toward each other by the angles α_{32} and α_{23}, respectively (Fig. 13). The change in angle (in the plane) between $\mathbf{A}$ and $\mathbf{B}$ due to the deformation is then the sum of these two angles:

$$\alpha_{32} + \alpha_{23} = 2\varepsilon_{23} \tag{2.48}$$

from Eq. 2.40. Note that $2\varepsilon_{23}$ is equivalent to the engineering shear strain component γ_{23}. [Although there are also out-of-plane (x_1) components of $\mathbf{A}'$ and $\mathbf{B}'$, such components are higher order and can be neglected if we consider only *infinitesimal* deformations. That is, the effects of these components on the rotation angle in the $x_2 - x_3$ plane are small.]

To summarize this discussion, the diagonal components of strain represent changes of length per unit length, and the off-diagonal components represent changes in angle (due to shearing):

$$\varepsilon_{11} = \frac{\delta A_1}{A_1}, \ \varepsilon_{22} = \frac{\delta A_2}{A_2}, \ \varepsilon_{33} = \frac{\delta A_3}{A_3}, \ 2\varepsilon_{ij} = \gamma_{ij} \ (i \neq j). \tag{2.49}$$

2.4 Strain as a Tensor

We now prove that the strain ε_{ij} is a tensor of rank 2. From Eq. 2.43,

$$A\,\delta A = \varepsilon_{ij}A_iA_j = \mathbf{A}^T\boldsymbol{\varepsilon}\mathbf{A}, \tag{2.50}$$

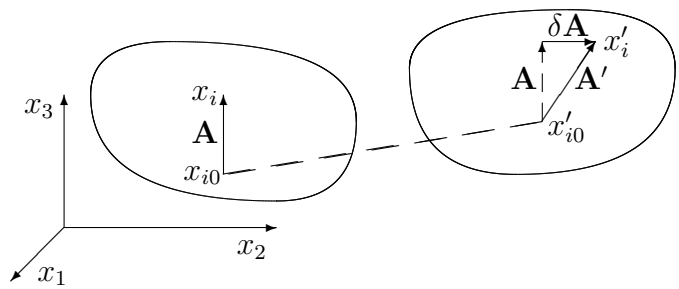


Figure 14: General Infinitesimal Transformation.

where A is the original length of the vector $\mathbf{A}$, and δA is the change in length of $\mathbf{A}$. The product $A\,\delta A$ is independent of the coordinate system (i.e., it is invariant under an orthogonal coordinate transformation). Thus, in two different coordinate systems,

$$\bar{\mathbf{A}}^T\bar{\boldsymbol{\varepsilon}}\bar{\mathbf{A}} = \mathbf{A}^T\boldsymbol{\varepsilon}\mathbf{A}. \tag{2.51}$$

However, since the vector $\mathbf{A}$ transforms according to the rule $\bar{\mathbf{A}} = \mathbf{R}\mathbf{A}$,

$$\bar{\mathbf{A}}^T\bar{\boldsymbol{\varepsilon}}\bar{\mathbf{A}} = (\mathbf{R}^T\bar{\mathbf{A}})^T\boldsymbol{\varepsilon}(\mathbf{R}^T\bar{\mathbf{A}}) = \bar{\mathbf{A}}^T\left(\mathbf{R}\boldsymbol{\varepsilon}\mathbf{R}^T\right)\bar{\mathbf{A}}. \tag{2.52}$$

Thus,

$$\bar{\boldsymbol{\varepsilon}} = \mathbf{R}\boldsymbol{\varepsilon}\mathbf{R}^T \quad \text{or} \quad \bar{\varepsilon}_{ij} = R_{ik}R_{jl}\varepsilon_{kl}. \tag{2.53}$$

That is, strain $\boldsymbol{\varepsilon}$ transforms like a tensor of rank 2, and the assertion is proved.

2.5 General Infinitesimal Deformation

Recall the affine transformation, where the components of the rotation/stretch vector are

$$\delta A_i = \alpha_{ij}A_j \quad (i = 1, 2, 3), \tag{2.54}$$

where δA_i are the components of the vector $\delta\mathbf{A}$.

Consider two nearby points x_{i0} and x_i in the undeformed configuration. The components of displacement of the first point (with coordinates x_{i0}) are (Fig. 14).

$$u_{i0} = x_{i0}' - x_{i0}. \tag{2.55}$$

The components of displacement of the second point (with coordinates x_i) are

$$u_i = x_i' - x_i. \tag{2.56}$$

Then,

$$\delta A_i = A_i' - A_i = (x_i' - x_{i0}') - (x_i - x_{i0}) = (x_i' - x_i) - (x_{i0}' - x_{i0}) = u_i - u_{i0}. \tag{2.57}$$

We now recall the Taylor series expansions with different numbers of variables. In one variable,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots. \tag{2.58}$$

In two variables,

$$f(x,y) = f(x_0,y_0) + \frac{\partial f(x_0,y_0)}{\partial x}(x-x_0) + \frac{\partial f(x_0,y_0)}{\partial y}(y-y_0) + \cdots . \tag{2.59}$$

In many variables, using index notation,

$$f(x_i) = f(x_{i0}) + \frac{\partial f(x_{i0})}{\partial x_j}(x_j - x_{j0}) + \cdots . \tag{2.60}$$

In Eq. 2.60, if f represents the displacement u_i,

$$u_i = u_{i0} + \frac{\partial u_i}{\partial x_j}(x_j - x_{j0}) + \cdots = u_{i0} + u_{i,j}A_j + \cdots . \tag{2.61}$$

For small deformation, we can ignore the higher order terms of the Taylor expansion, and write

$$u_i - u_{i0} = u_{i,j}A_j. \tag{2.62}$$

The left-hand side of this equation is given by Eq. 2.57, implying

$$\delta A_i = u_{i,j}A_j. \tag{2.63}$$

A comparison of this equation with Eq. 2.54 yields $\alpha_{ij} = u_{i,j}$. Thus, from Eqs. 2.39, 2.40, and 2.41,

$$u_{i,j} = \varepsilon_{ij} + \omega_{ij}, \tag{2.64}$$

where ε_{ij}, the strain tensor, is symmetric and given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.65}$$

and ω_{ij}, the rigid body rotation, is skew-symmetric and given by

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}). \tag{2.66}$$

Eq. 2.65 is referred to as the *strain-displacement equation.*

For further clarification, we write the strain-displacement equations in expanded notation, where we let (x,y,z) rather than (x_1,x_2,x_3) denote the coordinates, and we let (u,v,w) rather than (u_1,u_2,u_3) denote the displacement components. Thus, for a general three-dimensional body undergoing the deformation

$$u = u(x,y,z),\ v = v(x,y,z),\ w = w(x,y,z), \tag{2.67}$$

the strain-displacement equations are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x},\ \varepsilon_{yy} = \frac{\partial v}{\partial y},\ \varepsilon_{zz} = \frac{\partial w}{\partial z}, \tag{2.68}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\gamma_{xy}, \tag{2.69}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\gamma_{yz}, \tag{2.70}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\gamma_{xz}. \tag{2.71}$$

The components of strain in Eq. 2.68 are referred to as the *normal* (or *direct*) strains. The components of strain in Eqs. 2.69, 2.70, and 2.71 are referred to as the *shear* strains. The strains γ_{ij} are the engineering shear strains.

The main reason for defining mathematical shear strains which are half the engineering shear strains is that ε_{ij} is a tensor of rank 2.

2.6 Compatibility Equations

Given any arbitrary displacement field $\mathbf{u}(x_1, x_2, x_3)$, strains can be computed directly from the strain-displacement equations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.72}$$

However, the reverse is not true. The question is: Given any arbitrary set of strains, can the displacements be computed? Clearly, since rigid body translations and rotations do not contribute to strain, the best we could expect is uniqueness to arbitrary rigid body motions. We might anticipate a problem in solving Eq. 2.72 for the displacements, since Eq. 2.72 represents a system of six equations in three unknowns (u_1, u_2, u_3).

If we multiply each side of Eq. 2.72 by the alternating symbol e_{rip} and differentiate, we obtain

$$2e_{rip}\varepsilon_{ij,p} = e_{rip}(u_{i,j} + u_{j,i})_{,p} = e_{rip}u_{i,jp} + e_{rip}u_{j,ip}, \tag{2.73}$$

where the last term vanishes, since if i and p are interchanged, the derivative is unchanged, but e_{rip} changes signs. For example,

$$e_{r12}u_{j,12} + e_{r21}u_{j,21} = (e_{r12} + e_{r21})u_{j,12} = (e_{r12} - e_{r12})u_{j,12} = 0. \tag{2.74}$$

Thus, Eq. 2.73 becomes

$$2e_{rip}\varepsilon_{ij,p} = e_{rip}u_{i,jp}. \tag{2.75}$$

We now multiply both sides of this equation by e_{sjq} and differentiate to obtain

$$2e_{rip}e_{sjq}\varepsilon_{ij,pq} = e_{rip}e_{sjq}u_{i,jpq}, \tag{2.76}$$

where the right-hand side vanishes for the same reason that the last term in Eq. 2.73 vanishes. In this case, subscripts j and q are common to both the alternating symbol and the derivative. Thus,

$$e_{rip}e_{sjq}\varepsilon_{ij,pq} = 0. \tag{2.77}$$

These equations, known as the *compatibility equations*, are *necessary* for the strain-displacement equations to have a displacement solution.

The compatibility equations, Eq. 2.77, represent nine equations, since there are two free indices (r and s), each of which takes the values $1, 2, 3$. However, it turns out that, due to symmetries on ε_{ij} and on the derivatives, only six of the nine equations are unique.

Notice that, for each r and s, each equation has four terms, since, for fixed r and s, there are two nonzero choices for ip and two for jq. For example, if $r = s = 1$, Eq. 2.77 is

$$e_{123}\, e_{123}\, \varepsilon_{22,33} + e_{123}\, e_{132}\, \varepsilon_{23,32} + e_{132}\, e_{123}\, \varepsilon_{32,23} + e_{132}\, e_{132}\, \varepsilon_{33,22} = 0 \tag{2.78}$$

or

$$\varepsilon_{22,33} - \varepsilon_{23,32} - \varepsilon_{32,23} + \varepsilon_{33,22} = 0, \tag{2.79}$$

where the second and third terms are equal. Thus, the first of the six compatibility equations is

$$2\varepsilon_{32,23} = \varepsilon_{22,33} + \varepsilon_{33,22}. \tag{2.80}$$

The other five compatibility equations are

$$2\varepsilon_{12,12} = \varepsilon_{11,22} + \varepsilon_{22,11}, \tag{2.81}$$

$$2\varepsilon_{13,13} = \varepsilon_{11,33} + \varepsilon_{33,11}, \tag{2.82}$$

$$\varepsilon_{11,23} = -\varepsilon_{23,11} + \varepsilon_{31,12} + \varepsilon_{12,13}, \tag{2.83}$$

$$\varepsilon_{22,13} = -\varepsilon_{31,22} + \varepsilon_{12,23} + \varepsilon_{23,12}, \tag{2.84}$$

$$\varepsilon_{33,12} = -\varepsilon_{12,33} + \varepsilon_{23,31} + \varepsilon_{31,32}. \tag{2.85}$$

It can be shown that the above six equations can also be written in index notation as

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0. \tag{2.86}$$

The compatibility equations derived above have been shown to be *necessary* for the existence of a displacement field given the strains. It can be shown[18] that they are also *sufficient*. Hence, the compatibility equations are both necessary and sufficient for the existence of a displacement field, given the strains.

Two other properties of displacement fields result from the proof (not shown here) of the sufficiency of the compatibility equations:

1. Zero strains imply a rigid body displacement field (translation and rotation).

2. Given compatible strains, the displacements are unique to within a rigid body motion.

Neither of these properties is unexpected.

Note that, in two dimensions, there is only one non-trivial compatibility equation (Eq. 2.81).

2.7 Integrating the Strain-Displacement Equations

The strain compatibility equations are both necessary and sufficient for the existence of the displacement field, given the strains. Finding the displacements, however, requires integrating the strain-displacement equations.

Consider, for example, the two-dimensional strain field

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = A, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \quad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \tag{2.87}$$

where A is a specified constant. This strain field satisfies the compatibility equations. Integrating the first two of these equations yields

$$\begin{cases} u = Ax + f(y) \\ v = g(x), \end{cases} \tag{2.88}$$

where $f(y)$ and $g(x)$ are unknown functions of y and x, respectively. The substitution of these functions for u and v into Eq. 2.87c yields

$$f'(y) + g'(x) = 0 \tag{2.89}$$

or

$$f'(y) = -g'(x), \tag{2.90}$$

which states that a function of y equals a function of x for all x, y. Hence, each function must necessarily be a constant:

$$f'(y) = -g'(x) = B. \tag{2.91}$$

Thus,

$$\begin{cases} f(y) = By + C \\ g(x) = -Bx + D, \end{cases} \tag{2.92}$$

where C and D and constants, and

$$\begin{cases} u = Ax + By + C \\ v = -Bx + D. \end{cases} \tag{2.93}$$

We can write this result in matrix form involving symmetric and antisymmetric matrices:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} C \\ D \end{Bmatrix}, \tag{2.94}$$

where the first term on the right-hand side represents the stretch, the second term represents rigid body rotation, and the third term represents rigid body translation. In two dimensions, rigid body motion has three degrees of freedom (B, C, D). Notice that the second matrix on the right-hand side is skew-symmetric, as required for a rotation.

2.8 Principal Axes of Strain

Consider the strain tensor

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}. \tag{2.95}$$

A key problem is to determine whether there is an orthogonal transformation of coordinates (i.e., a rotation of axes) $\mathbf{x}' = \mathbf{R}\mathbf{x}$ such that the strain tensor $\boldsymbol{\varepsilon}$ is diagonalized:

$$\boldsymbol{\varepsilon}' = \begin{bmatrix} \varepsilon'_{11} & 0 & 0 \\ 0 & \varepsilon'_{22} & 0 \\ 0 & 0 & \varepsilon'_{33} \end{bmatrix} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}, \tag{2.96}$$

where the diagonal strains in this new coordinate system are denoted e_1, e_2, e_3.

Since the strain $\boldsymbol{\varepsilon}$ is a tensor of rank two,

$$\mathbf{R}\boldsymbol{\varepsilon}\mathbf{R}^T = \boldsymbol{\varepsilon}' \quad \text{or} \quad \boldsymbol{\varepsilon}\mathbf{R}^T = \mathbf{R}^T\boldsymbol{\varepsilon}'. \tag{2.97}$$

We now let $\mathbf{v}_i$ denote the ith column of $\mathbf{R}^T$; i.e.,

$$\mathbf{R}^T = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3], \tag{2.98}$$

in which case Eq. 2.97 can be written in matrix form as

$$\boldsymbol{\varepsilon}[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix} = [e_1\mathbf{v}_1 \ \ e_2\mathbf{v}_2 \ \ e_3\mathbf{v}_3]. \tag{2.99}$$

(The various matrices in this equation are conformable from a "block" point of view, since the left-hand side is, for example, the product of a 1×1 matrix with a 1×3 matrix.) Each column of this equation is

$$\boldsymbol{\varepsilon}\mathbf{v}_i = e_i\mathbf{v}_i \quad \text{(no sum on } i\text{)}. \tag{2.100}$$

Thus, the original desire to find a coordinate rotation which would transform the strain tensor to diagonal form reduces to seeking vectors $\mathbf{v}$ such that

$$\boldsymbol{\varepsilon}\mathbf{v} = e\mathbf{v}. \tag{2.101}$$

Eq. 2.101 is an *eigenvalue* problem with e the eigenvalue, and $\mathbf{v}$ the corresponding eigenvector. The goal in solving Eq. 2.101 (and eigenvalue problems in general) is to find nonzero vectors $\mathbf{v}$ which satisfy Eq. 2.101 for some scalar e. Geometrically, the goal in solving Eq. 2.101 is to find nonzero vectors $\mathbf{v}$ which, when multiplied (or transformed) by the matrix $\boldsymbol{\varepsilon}$, result in new vectors which are parallel to $\mathbf{v}$.

Eq. 2.101 is equivalent to the matrix system

$$(\boldsymbol{\varepsilon} - e\mathbf{I})\mathbf{v} = \mathbf{0}. \tag{2.102}$$

This equation clearly has the trivial solution $\mathbf{v} = \mathbf{0}$. Indeed, if the left-hand side matrix $\boldsymbol{\varepsilon} - e\mathbf{I}$ were nonsingular, the only solution would be $\mathbf{v} = \mathbf{0}$. Thus, for nontrivial solutions ($\mathbf{v} \neq \mathbf{0}$), the matrix $\boldsymbol{\varepsilon} - e\mathbf{I}$ must be singular, implying that

$$\det(\boldsymbol{\varepsilon} - e\mathbf{I}) = \begin{vmatrix} \varepsilon_{11} - e & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - e & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - e \end{vmatrix} = 0, \tag{2.103}$$

which is referred to as the *characteristic equation* associated with the eigenvalue problem. The characteristic equation is a cubic polynomial in e, since, when expanded, yields

$$\begin{aligned}(\varepsilon_{11} - e)[(\varepsilon_{22} - e)(\varepsilon_{33} - e) - \varepsilon_{23}\varepsilon_{32}] - \varepsilon_{12}[\varepsilon_{21}(\varepsilon_{33} - e) - \varepsilon_{23}\varepsilon_{31}] \\ + \varepsilon_{13}[\varepsilon_{21}\varepsilon_{32} - \varepsilon_{31}(\varepsilon_{22} - e)] = 0\end{aligned} \tag{2.104}$$

or

$$-e^3 + \theta_1 e^2 - \theta_2 e + \theta_3 = 0, \tag{2.105}$$

where

$$\begin{cases} \theta_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{ii} = \operatorname{tr} \boldsymbol{\varepsilon} \\ \theta_2 = \varepsilon_{22}\varepsilon_{33} + \varepsilon_{33}\varepsilon_{11} + \varepsilon_{11}\varepsilon_{22} - \varepsilon_{31}^2 - \varepsilon_{12}^2 - \varepsilon_{23}^2 = \frac{1}{2}(\varepsilon_{ii}\varepsilon_{jj} - \varepsilon_{ij}\varepsilon_{ji}) \\ \theta_3 = \det \boldsymbol{\varepsilon}. \end{cases} \tag{2.106}$$

The strains e_1, e_2, e_3, which are the three solutions of the characteristic equation, are referred to as the *principal strains*. The resulting strain tensor is

$$\boldsymbol{\varepsilon} = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{bmatrix}, \tag{2.107}$$

and the coordinate axes of the coordinate system in which the strain tensor is diagonal is referred to as the *principal axes of strain* or the *principal coordinates*.

The principal axes of strain are the eigenvectors of the strain tensor, since the eigenvectors (if scaled to unit length) are columns of $\mathbf{R}^T$ (rows of $\mathbf{R}$), where

$$R_{ij} = \mathbf{e}_i' \cdot \mathbf{e}_j. \tag{2.108}$$

Thus, Row i of $\mathbf{R}$ consists of the direction cosines of the ith principal axis. Although, in general, eigenvectors can have any scaling, the eigenvectors must be scaled to unit length when placed in the rotation matrix $\mathbf{R}$; otherwise, $\mathbf{R}$ would not be orthogonal.

Since the principal strains are independent of the original coordinate system, the coefficients of the characteristic polynomial, Eq. 2.105, must be *invariant* with respect to a coordinate rotation. Thus, θ_1, θ_2, and θ_3 are referred to as the *invariants* of the strain tensor. That is, the three invariants have the same values in all coordinate systems.

In principal coordinates, the strain invariants are

$$\begin{cases} \theta_1 = e_1 + e_2 + e_3 \\ \theta_2 = e_2 e_3 + e_3 e_1 + e_1 e_2 \\ \theta_3 = e_1 e_2 e_3 \end{cases} \tag{2.109}$$

We note that the above theory for eigenvalues, principal axes, and invariants is applicable to all tensors of rank 2, since we did not use the fact that we were dealing specifically with strain. For example, stress (which has not been discussed yet) will also turn out to be a tensor of rank 2. A geometrical example of a tensor of rank 2 is the inertia matrix I_{ij} whose matrix elements are moments of inertia. The determination of principal axes of inertia in two dimensions in an eigenvalue problem.

What we have proved here in the context of strain is one of the fundamental results of linear algebra: Given a real, symmetric matrix $\mathbf{A}$, the matrix product

$$\mathbf{S}^T\mathbf{A}\mathbf{S} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \tag{2.110}$$

is diagonal, where $\mathbf{S}$ is the matrix whose columns are the normalized eigenvectors, and λ_i is the ith eigenvalue. Thus, $\mathbf{S}$ is the diagonalizing matrix for $\mathbf{A}$. More generally, if the matrix $\mathbf{A}$ is unsymmetric, and the eigenvectors of $\mathbf{A}$ are linearly independent, and a matrix $\mathbf{S}$ is formed whose columns are the eigenvectors of $\mathbf{A}$ (with any normalization), then the matrix product

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} \tag{2.111}$$

is diagonal. This property is proved in block form as follows: If $\mathbf{x}^{(i)}$ is the ith eigenvector of $\mathbf{A}$, then

$$\begin{aligned} \mathbf{A}\mathbf{S} &= \mathbf{A}\begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \cdots & \mathbf{x}^{(n)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{A}\mathbf{x}^{(1)} & \mathbf{A}\mathbf{x}^{(2)} & \cdots & \mathbf{A}\mathbf{x}^{(n)} \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\mathbf{x}^{(1)} & \lambda_2\mathbf{x}^{(2)} & \cdots & \lambda_n\mathbf{x}^{(n)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \cdots & \mathbf{x}^{(n)} \end{bmatrix} \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} = \mathbf{S}\mathbf{\Lambda}. \end{aligned} \tag{2.112}$$

Since the columns of $\mathbf{S}$ are independent, $\mathbf{S}$ is invertible, and

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{\Lambda}. \tag{2.113}$$

2.9 Properties of the Real Symmetric Eigenvalue Problem

Since the matrix coefficients in the principal strain eigenvalue problem, Eq. 2.101, are real and symmetric, the eigenvalue problem is termed a real symmetric eigenvalue problem. We can write such problems in the general form

$$\mathbf{M}\mathbf{x} = \lambda\mathbf{x}, \tag{2.114}$$

where the coefficient matrix $\mathbf{M}$ is real and symmetric, and λ denotes the eigenvalue. Real symmetric eigenvalue problems have two additional general properties of interest.

Property 1. For the real symmetric matrix $\mathbf{M}$, the eigenvalue problem, Eq. 2.114, has real eigenvalues. To prove this statement, we first take the complex conjugate of both sides of this equation to obtain

$$\mathbf{M}\mathbf{x}^* = \lambda^*\mathbf{x}^*, \tag{2.115}$$

where the asterisk denotes the complex conjugate. If we multiply Eq. 2.114 by $\mathbf{x}^{*T}$ and Eq. 2.115 by $\mathbf{x}^T$, we obtain

$$\mathbf{x}^{*T}\mathbf{M}\mathbf{x} = \lambda\mathbf{x}^{*T}\mathbf{x} \quad \text{and} \quad \mathbf{x}^T\mathbf{M}\mathbf{x}^* = \lambda^*\mathbf{x}^T\mathbf{x}^*. \tag{2.116}$$

The left-hand sides of these two equations are scalars and hence equal, since a scalar is equal to its own transpose. Thus, by equating the two right-hand sides, we obtain

$$(\lambda - \lambda^*)\mathbf{x}^{*T}\mathbf{x} = (\lambda - \lambda^*)|\mathbf{x}|^2 = 0. \tag{2.117}$$

Since $\mathbf{x} \neq \mathbf{0}$, $\lambda^* = \lambda$. That is, λ is real, and the first property is proved.

Property 2. For the real symmetric matrix $\mathbf{M}$, the eigenvalue problem, Eq. 2.114, has orthogonal eigenvectors if the eigenvalues are distinct. To prove this statement, we first consider two eigenvalue/eigenvector pairs for distinct eigenvalues λ_1 and λ_2, in which case

$$\mathbf{M}\mathbf{x}_1 = \lambda_1\mathbf{x}_1, \quad \mathbf{M}\mathbf{x}_2 = \lambda_2\mathbf{x}_2. \tag{2.118}$$

If we now multiply the first of these equations by $\mathbf{x}_2^T$, the second by $\mathbf{x}_1^T$, and subtract, we obtain

$$(\lambda_1 - \lambda_2)\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_2^T\mathbf{M}\mathbf{x}_1 - \mathbf{x}_1^T\mathbf{M}\mathbf{x}_2, \tag{2.119}$$

where we note that, for two vectors $\mathbf{a}$ and $\mathbf{b}$, the matrix product $\mathbf{a}^T\mathbf{b}$ is equivalent to the dot product $\mathbf{a} \cdot \mathbf{b}$. The first term on the right-hand side is

$$\mathbf{x}_2^T\mathbf{M}\mathbf{x}_1 = (\mathbf{x}_2^T\mathbf{M}\mathbf{x}_1)^T = \mathbf{x}_1^T\mathbf{M}^T\mathbf{x}_2 = \mathbf{x}_1^T\mathbf{M}\mathbf{x}_2, \tag{2.120}$$

where the first equation follows since a scalar is "symmetric" (and is equal to its own transpose), the second equation follows since the transpose of a matrix product is equal to the product of the individual transposes in reverse order, and the third equation follows from the symmetry of $\mathbf{M}$. Thus, from Eq. 2.119,

$$(\lambda_1 - \lambda_2)\mathbf{x}_1 \cdot \mathbf{x}_2 = 0, \tag{2.121}$$

and, for distinct eigenvalues ($\lambda_1 \neq \lambda_2$), $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$. Thus the principal directions associated with a matrix are mutually orthogonal.

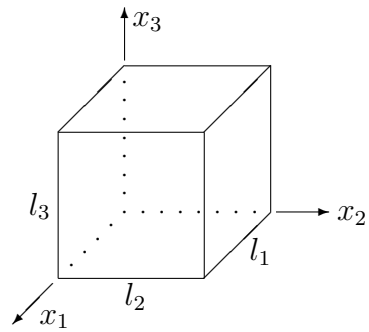


Figure 15: Rectangular Parallelepiped.

2.10 Geometrical Interpretation of the First Invariant

Recall from Eq. 2.106 that the first invariant of the strain tensor is

$$\theta_1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = \varepsilon_{ii} = \text{tr } \boldsymbol{\varepsilon}. \tag{2.122}$$

Consider a rectangular parallelepiped of sides l_1, l_2, l_3 (Fig. 15). The original volume V of the parallelepiped (before deformation) is $V = l_1 l_2 l_3$. Since shear strains do not cause a change in volume (e.g., see Fig. 13), the new volume after straining is given by

$$V + \Delta V = l_1(1+\varepsilon_{11})l_2(1+\varepsilon_{22})l_3(1+\varepsilon_{33}) = l_1 l_2 l_3(1+\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}+\cdots), \tag{2.123}$$

where the terms omitted from the right-hand side involve products of strains which can be ignored if the strains are infinitesimal. Thus,

$$\Delta V = l_1 l_2 l_3(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) = V\varepsilon_{ii} \tag{2.124}$$

or

$$\varepsilon_{ii} = \frac{\Delta V}{V}, \tag{2.125}$$

which is consistent with Eq. 2.18. The first invariant (equal to the trace of the strain tensor) is referred to as the *volumetric strain*.

2.11 Finite Deformation

The strain-displacement relations derived in §2.5 made the assumption of small deformations. Here we consider a more general derivation of the strain-displacement relations which would be needed for a variety of problems involving large deformations and other nonlinear effects such as buckling.

Let x_i denote the Cartesian coordinates of a point in an undeformed body (Fig. 16). Let x_i' denote the coordinates of the same point in the deformed body. The displacement of the point is

$$u_i = x_i' - x_i. \tag{2.126}$$

We let $d\ell$ and $d\ell'$ denote the distance between x_i and a nearby point in the body before and after deformation, respectively. Then,

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i\, dx_i. \tag{2.127}$$

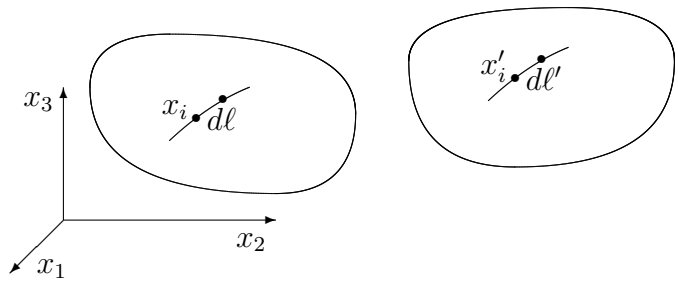


Figure 16: Geometry of Finite Deformation.

Similarly, for the deformed body,

$$d\ell'^2 = dx'_i\, dx'_i. \tag{2.128}$$

From Eq. 2.126,

$$dx'_i = dx_i + du_i, \tag{2.129}$$

where, from the chain rule,

$$du_i = \frac{\partial u_i}{\partial x_j}\, dx_j = u_{i,j}\, dx_j, \tag{2.130}$$

since the displacements u_i depend on the coordinates. Then,

$$\begin{aligned} d\ell'^2 &= (dx_i + u_{i,j}\, dx_j)(dx_i + u_{i,k}\, dx_k) && (2.131)\\ &= dx_i\, dx_i + u_{i,j}\, dx_i\, dx_j + u_{i,k}\, dx_i\, dx_k + u_{i,j}u_{i,k}\, dx_j\, dx_k && (2.132)\\ &= d\ell^2 + u_{i,j}\, dx_i\, dx_j + u_{j,i}\, dx_i\, dx_j + u_{k,i}u_{k,j}\, dx_i\, dx_j, && (2.133) \end{aligned}$$

where, in the third term, we have replaced i by j and k by i, and, in the last term, we have interchanged i and k. Hence,

$$d\ell'^2 - d\ell^2 = (u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})\, dx_i\, dx_j. \tag{2.134}$$

We take the difference $d\ell'^2 - d\ell^2$ as the measure of strain and *define* ε_{ij} by

$$d\ell'^2 - d\ell^2 = 2\varepsilon_{ij}\, dx_i\, dx_j. \tag{2.135}$$

Thus,

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}). \tag{2.136}$$

For example,

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \tag{2.137}$$

and

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right). \tag{2.138}$$

The other strain expressions are similar. Note that, for small strains, the nonlinear (product) terms can be neglected, and Eq. 2.136 reduces to Eq. 2.65.

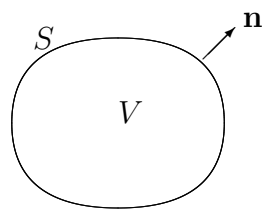


Figure 17: Arbitrary Body of Volume V and Surface S.

3 Stress

Consider a general three-dimensional elastic body (Fig. 17) acted upon by forces. There are two kinds of forces of interest:

1. body forces, which act at each point throughout the volume of the body (e.g., gravity), and
2. surface forces (tractions), which act only on surface points (e.g., hydrostatic pressure on a submerged body).

Let ρ denote the mass density (mass per unit volume) of the body material. The mass of the body is then

$$\int_V \rho \, dV.$$

The net body force due to gravity is

$$\int_V \rho \mathbf{g} \, dV,$$

where $\mathbf{g}$ is the acceleration due to gravity vector. In general, if $\mathbf{f}$ denotes the body force per unit mass, the net force on the body due to $\mathbf{f}$ is

$$\int_V \rho \mathbf{f} \, dV,$$

where $\rho\mathbf{f}$ is the body force per unit volume. The net moment due to the body force $\mathbf{f}$ is then

$$\int_V \mathbf{r} \times \rho \mathbf{f} \, dV,$$

where $\mathbf{r}$ is the *position vector* given in Cartesian coordinates by

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = x_i\mathbf{e}_i. \tag{3.1}$$

The position vector of a point is thus the vector from the coordinate origin to the point.

Let $\mathbf{t}(\mathbf{x}, \mathbf{n})$ denote the *stress vector*, the internal force per unit area on an internal surface (at a point $\mathbf{x}$) with normal $\mathbf{n}$ (Fig. 18). This vector depends on the orientation of the surface and has three components. On an external surface, we denote $\mathbf{t}$ by $\mathbf{T}$, which is referred to as the vector of *surface tractions*. The force on the surface S is then

$$\oint_S \mathbf{t} \, dS,$$

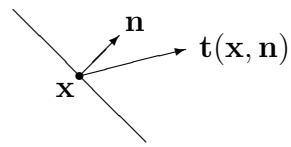


Figure 18: Stress Vector.

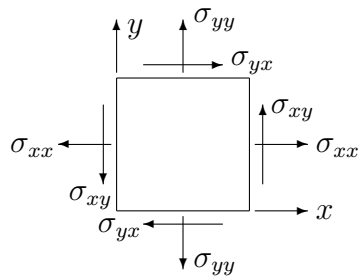


Figure 19: Sign Conventions for Stress Components.

where S may be either an internal or external surface. If we consider all forces (surface and body), the net force on the volume V is

$$\int_V \rho \mathbf{f}\, dV + \oint_S \mathbf{t}\, dS.$$

Similarly, the net moment on V is

$$\int_V \mathbf{r} \times \rho \mathbf{f}\, dV + \oint_S \mathbf{r} \times \mathbf{t}\, dS.$$

Since $\mathbf{t}(\mathbf{x}, \mathbf{n})$ at a point $\mathbf{x}$ depends on the surface in question (with normal $\mathbf{n}$) and has three components, we define $t_j(\mathbf{x}, \mathbf{e}_i) = \sigma_{ij}$ as the jth component of the stress vector acting on a surface with normal in the ith coordinate direction. That is,

$$\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}(\mathbf{x}, \mathbf{e}_i), \tag{3.2}$$

where, in three dimensions, i and j take the values $1, 2, 3$.

We use the following sign conventions for the stress components (Fig. 19):

1. A component of normal stress (σ_{11}, σ_{22}, σ_{33}) is positive in tension (i.e., $\mathbf{t}$ acts away from the surface to which it is applied).

2. A component of shear stress (σ_{12}, σ_{13}, σ_{21}, σ_{23}, σ_{31}, σ_{32}) is positive if it acts in the positive coordinate direction on a face where the normal stress acts in the positive coordinate direction.

We wish to relate the stress vector $\mathbf{t}$ to the stress components σ_{ij}. Consider a small trirectangular tetrahedron for which the inclined face has normal $\mathbf{n}$ (Fig. 20). A trirectangular tetrahedron is a tetrahedron with three right triangular faces. Let h denote the "height" of the tetrahedron (i.e., the distance from the inclined plane to the origin), ΔS denote the

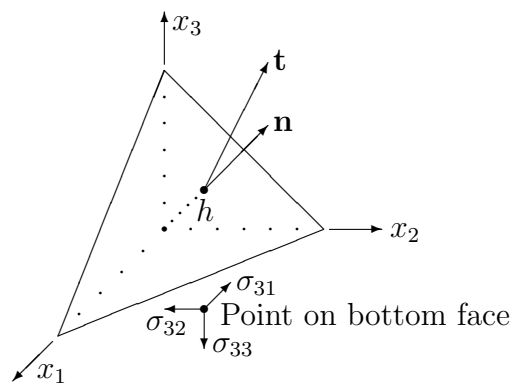


Figure 20: Stresses on Small Tetrahedron.

area of the inclined triangle, and ΔS_i denote the area of the right triangle in the coordinate plane having $\mathbf{e}_i$ as its unit normal. In general, according to de Gua's theorem [J.P. de Gua de Malves (1712-1785)],

$$\Delta S_i = n_i \, \Delta S, \tag{3.3}$$

where n_i is the ith component of the normal $\mathbf{n}$ (i.e., the direction cosines). Note, from geometry, that the volume of the tetrahedron is $(\Delta S)h/3$.

For a small volume (small h), the body force will not vary much over the volume. Hence, the force balance in the ith direction is

$$(\rho f_i)V + (\Delta S)t_i - \sigma_{1i} \, \Delta S_1 - \sigma_{2i} \, \Delta S_2 - \sigma_{3i} \, \Delta S_3 = 0, \tag{3.4}$$

where the first term is the body force, the second term is the force on the inclined surface, and the last three terms are the forces on the three coordinate surfaces. Thus,

$$\rho f_i h(\Delta S)/3 + t_i(\Delta S) - \sigma_{ji}(\Delta S_j) = 0, \tag{3.5}$$

where $\Delta S_j = n_j \, \Delta S$ from Eq. 3.3. Hence

$$\rho f_i h/3 + t_i - \sigma_{ji} n_j = 0. \tag{3.6}$$

If we now take the limit as $h \to 0$, the first term disappears, and

$$t_i = \sigma_{ji} n_j \tag{3.7}$$

That is, we have expressed the stress vector at a point on a surface in terms of the individual components of stress at that location and the surface normal.

3.1 Momentum Equation

To derive the equations of dynamic equilibrium for a body, we use Newton's second law, which states that the net force on a body (the sum of surface and body forces) equals the product of mass and acceleration:

$$\oint_S \mathbf{t} \, dS + \int_V \rho \mathbf{f} \, dV = \int_V \ddot{\mathbf{u}} \, \rho \, dV \tag{3.8}$$

or, in index notation,

$$\oint_S t_i \, dS + \int_V \rho f_i \, dV = \int_V \rho \ddot{u}_i \, dV, \tag{3.9}$$

where dots denote time derivatives. The first term of this equation (the surface traction) is given by

$$\oint_S t_i \, dS = \oint_S \sigma_{ji} n_j \, dS = \int_V \sigma_{ji,j} \, dV, \tag{3.10}$$

where the second equality follows from the divergence theorem, Eq. 1.90. Thus, from Eq. 3.9,

$$\int_V (\sigma_{ji,j} + \rho f_i - \rho \ddot{u}_i) \, dV = 0. \tag{3.11}$$

Since this equation must hold for all V, the integrand must vanish at points of continuity of the integrand:

$$\sigma_{ji,j} + \rho f_i = \rho \ddot{u}_i, \tag{3.12}$$

which is referred to as the *momentum equation*, since it arose from the balance of linear momenta.

3.2 Angular Momentum

The net moment of the forces acting on a body is equal to the rate of change of the moment of momentum (angular momentum):

$$\int_V \mathbf{r} \times \rho \mathbf{f} \, dV + \oint_S \mathbf{r} \times \mathbf{t} \, dS = \frac{d}{dt} \int_V \mathbf{r} \times \rho \dot{\mathbf{u}} \, dV, \tag{3.13}$$

where $\mathbf{r}$ is the position vector given by

$$\mathbf{r} = x_i \mathbf{e}_i. \tag{3.14}$$

Recall from Eq. 1.25 that, for two vectors $\mathbf{a}$ and $\mathbf{b}$, the kth component of the cross product is

$$(\mathbf{a} \times \mathbf{b})_k = a_i b_j e_{ijk}, \tag{3.15}$$

where e_{ijk} is the alternating symbol. Then, from Eq. 3.13, in index notation,

$$\int_V \rho x_i f_j e_{ijk} \, dV + \oint_S x_i t_j e_{ijk} \, dS = \int_V \rho x_i \ddot{u}_j e_{ijk} \, dV, \tag{3.16}$$

where, from Eq. 3.7, $t_j = \sigma_{lj} n_l$. Thus, from the divergence theorem,

$$\int_V \rho x_i f_j e_{ijk} \, dV + \int_V (x_i \sigma_{lj} e_{ijk})_{,l} \, dV = \int_V \rho x_i \ddot{u}_j e_{ijk} \, dV. \tag{3.17}$$

By expanding the derivative in the second integral, we obtain

$$\int_V \rho x_i f_j e_{ijk} \, dV + \int_V (x_{i,l} \sigma_{lj} e_{ijk} + x_i \sigma_{lj,l} e_{ijk}) \, dV = \int_V \rho x_i \ddot{u}_j e_{ijk} \, dV, \tag{3.18}$$

where $x_{i,l} = \delta_{il}$, so that

$$\int_V \left[x_i \left(\sigma_{lj,l} + \rho f_j - \rho \ddot{u}_j\right) e_{ijk} + \sigma_{ij} e_{ijk}\right] dV = 0. \tag{3.19}$$

Since, by the momentum equation, Eq. 3.12, the parenthetical expression vanishes, we obtain

$$\int_V \sigma_{ij} e_{ijk} \, dV = 0, \tag{3.20}$$

which must hold *for all* V (i.e., V is arbitrary). Thus, at points of continuity of the integrand, the integrand must vanish, and

$$\sigma_{ij} e_{ijk} = 0 \quad (k = 1, 2, 3). \tag{3.21}$$

For example, for $k = 3$,

$$0 = \sigma_{12} e_{123} + \sigma_{21} e_{213} = \sigma_{12}(1) + \sigma_{21}(-1) \tag{3.22}$$

or

$$\sigma_{12} = \sigma_{21} \tag{3.23}$$

and, similarly,

$$\sigma_{13} = \sigma_{31}, \ \sigma_{23} = \sigma_{32}. \tag{3.24}$$

Thus, in index notation, the balance of moments implies

$$\sigma_{ij} = \sigma_{ji}. \tag{3.25}$$

That is, $\boldsymbol{\sigma}$ is symmetric.

Since $\boldsymbol{\sigma}$ is symmetric, we can re-write the momentum and stress vector equations as

$$\begin{cases} \sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i, \\ t_i = \sigma_{ij} n_j \end{cases} \tag{3.26}$$

or $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$. In expanded form, the momentum equations are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho f_x = \rho \ddot{u}, \tag{3.27}$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho f_y = \rho \ddot{v}, \tag{3.28}$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho f_z = \rho \ddot{w}, \tag{3.29}$$

where the three Cartesian components of displacement are (u, v, w).

The momentum equation, Eq. 3.26a, reduces for statics problems to the equilibrium equation

$$\sigma_{ij,j} = -\rho f_i, \tag{3.30}$$

since the acceleration term is zero. In the absence of body forces (e.g., gravity), the equilibrium equation is simply

$$\sigma_{ij,j} = 0. \tag{3.31}$$

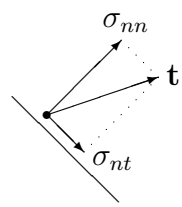


Figure 21: Components of Stress Vector.

3.3 Stress as a Tensor

We now show that $\boldsymbol{\sigma}$ is a tensor of rank 2. From Eq. 3.26b, the stress vector is

$$t_i = \sigma_{ij} n_j \quad \text{or} \quad \mathbf{t} = \boldsymbol{\sigma}\mathbf{n}. \tag{3.32}$$

In another coordinate system, $\hat{x}_i$,

$$\hat{\mathbf{t}} = \hat{\boldsymbol{\sigma}}\hat{\mathbf{n}}, \tag{3.33}$$

where the vectors transform under an orthogonal coordinate transformation according to the rule for tensors of rank 1:

$$\mathbf{t} = \mathbf{R}^T\hat{\mathbf{t}}, \ \mathbf{n} = \mathbf{R}^T\hat{\mathbf{n}}. \tag{3.34}$$

Thus,

$$\mathbf{R}^T\hat{\mathbf{t}} = \boldsymbol{\sigma}\mathbf{R}^T\hat{\mathbf{n}}. \tag{3.35}$$

If we multiply both sides of this equation by the orthogonal rotation matrix $\mathbf{R}$, we obtain

$$\hat{\mathbf{t}} = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T\hat{\mathbf{n}}. \tag{3.36}$$

since $\mathbf{R}\mathbf{R}^T = \mathbf{I}$. A comparison of this equation with Eq. 3.33 yields

$$\hat{\boldsymbol{\sigma}} = \mathbf{R}\boldsymbol{\sigma}\mathbf{R}^T \quad \text{or} \quad \hat{\sigma}_{ij} = R_{ik}\sigma_{kl}R^T_{lj} = R_{ik}R_{jl}\sigma_{kl}. \tag{3.37}$$

Thus, $\boldsymbol{\sigma}$ is a tensor of rank 2.

The stress tensor is symmetric and real. Thus, from the discussion of principal axes of strain in §2.8, the stress tensor also has three real eigenvalues (principal stresses) and three mutually orthogonal eigenvectors (the principal directions of stress). The eigenvalue problem to determine the principal stresses is thus

$$\boldsymbol{\sigma}\mathbf{x} = \bar{\sigma}\mathbf{x}, \tag{3.38}$$

where the eigenvalue is denoted $\bar{\sigma}$.

The stress vector equation, $\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}$, allows us to compute specific components of stress on a surface with normal $\mathbf{n}$. For example, in Fig. 21, if the stress tensor in Cartesian coordinates is $\boldsymbol{\sigma}$, the stress vector on the surface with normal $\mathbf{n}$ is $\boldsymbol{\sigma}\mathbf{n}$, and the normal stress on that surface is

$$\sigma_{nn} = \mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n} = \sigma_{ij} n_i n_j. \tag{3.39}$$

Pythagoras' theorem can then be used to deduce the shear stress on a face given the stress vector and the normal stress. Note also that this calculation of σ_{nn} is representative of the transformation of second-rank tensors in general. Thus, the normal component of strain on a surface with normal $\mathbf{n}$ is

$$\varepsilon_{nn} = \varepsilon_{ij} n_i n_j. \tag{3.40}$$

3.4 Mean Stress in a Deformed Body

Consider a body in a state of static stress with no body force. The equilibrium equation, which must hold at every point in a body, is

$$\sigma_{ij,j} = 0. \tag{3.41}$$

Thus,

$$\int_V \sigma_{ij,j} x_k \, dV = 0. \tag{3.42}$$

The integrand can be written in the form

$$\sigma_{ij,j} x_k = (\sigma_{ij} x_k)_{,j} - \sigma_{ij} x_{k,j} = (\sigma_{ij} x_k)_{,j} - \sigma_{ik}, \tag{3.43}$$

since $x_{k,j} = \delta_{kj}$. Thus,

$$\int_V (\sigma_{ij} x_k)_{,j} \, dV - \int_V \sigma_{ik} \, dV = 0. \tag{3.44}$$

The *mean stress* over the volume is defined as

$$\bar{\sigma}_{ik} = \frac{1}{V} \int_V \sigma_{ik} \, dV. \tag{3.45}$$

Hence,

$$\bar{\sigma}_{ik} = \frac{1}{V} \int_V (\sigma_{ij} x_k)_{,j} \, dV = \frac{1}{V} \oint_S \sigma_{ij} x_k n_j \, dS = \frac{1}{V} \oint_S t_i x_k \, dS, \tag{3.46}$$

where the second equation results from the divergence theorem. Since the stress tensor is symmetric, this formula can be written in the symmetric form

$$\bar{\sigma}_{ij} = \frac{1}{2V} \oint_S (t_i x_j + t_j x_i) \, dS. \tag{3.47}$$

Thus, the mean value of the stress tensor in a body with no body force can be found from the external forces (tractions) acting on the body without solving the equations of equilibrium.

3.5 Fluid-Structure Interface Condition

Consider an elastic structure in contact with a compressible, inviscid fluid (sometimes referred to as the "acoustic" fluid), for which the pressure satisfies the wave equation

$$\nabla^2 p = \frac{1}{c^2} \ddot{p}, \tag{3.48}$$

where c is the speed of sound in the fluid, and dots denote derivatives with respect to the time t. For exterior fluids, problems of interest include underwater vibrations, acoustic radiation and scattering, and shock. For interior fluids, problems of interest include the dynamics of acoustic cavities, fluid-filled tanks, and piping systems. If the fluid at the fluid-structure interface is non-cavitating, the interface condition follows immediately from the momentum equation (Eq. 3.26):

$$\frac{\partial p}{\partial n} = -\rho \ddot{u}_n, \tag{3.49}$$

where p is the fluid pressure (positive in compression) and $\ddot{u}_n$ is the normal component of acceleration at the interface. Since this fluid model cannot support shear, the only nonzero component of stress at the interface is the direct stress in the normal direction. In a coupled fluid-structure problem, the coupling is two-way. Thus, Eq. 3.49 can be viewed as the effect of the structure on the fluid. The effect of the fluid on the structure is simply that the normal stress must match the fluid pressure:

$$\sigma_n = -p. \tag{3.50}$$

4 Equations of Elasticity

4.1 Hooke's Law

The discussion of deformation involved only kinematics and geometry without the need to discuss the forces which might be involved in causing the deformation. On the other hand, the discussion of stress related the various forces to each other and determined the equations of equilibrium. Now we wish to relate the forces to the deformations.

Some materials (e.g., steel), when loaded, obey *Hooke's law*, which states that the extension is proportional to the force (or stress is proportional to strain). Thus, in uniaxial loading (one dimension),

$$\sigma = E\varepsilon, \tag{4.1}$$

where the proportionality constant E, which must be experimentally determined, is called *Young's modulus* or the *modulus of elasticity*.

In general three-dimensional elasticity, there are nine components of stress and nine components of strain (of which, due to symmetry, only six are unique). Thus, the three-dimensional generalization of Hooke's law is referred to as *generalized Hooke's law*, which states that each component of stress is a linear combination of all the strain components; i.e., in index notation,

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}, \tag{4.2}$$

where the elastic constants are denoted c_{ijkl}. For fixed ij, each of these equations has nine terms.

We now prove that c_{ijkl} is a tensor of rank 4. We can write Eq. 4.2 in terms of stress and strain in a second orthogonal coordinate system as

$$R_{ki}R_{lj}\bar{\sigma}_{kl} = c_{ijkl}R_{mk}R_{nl}\bar{\varepsilon}_{mn}. \tag{4.3}$$

If we multiply both sides of this equation by $R_{pj}R_{oi}$, and sum repeated indices, we obtain

$$R_{pj}R_{oi}R_{ki}R_{lj}\bar{\sigma}_{kl} = R_{oi}R_{pj}R_{mk}R_{nl}c_{ijkl}\bar{\varepsilon}_{mn}, \tag{4.4}$$

or, because $\mathbf{R}$ is an orthogonal matrix,

$$\delta_{ok}\delta_{pl}\bar{\sigma}_{kl} = \bar{\sigma}_{op} = R_{oi}R_{pj}R_{mk}R_{nl}c_{ijkl}\bar{\varepsilon}_{mn}. \tag{4.5}$$

Since, in the second coordinate system,

$$\bar{\sigma}_{op} = \bar{c}_{opmn}\bar{\varepsilon}_{mn}, \tag{4.6}$$

we conclude that

$$\bar{c}_{opmn} = R_{oi} R_{pj} R_{mk} R_{nl} c_{ijkl}, \tag{4.7}$$

which proves that c_{ijkl} is a tensor of rank 4.

Because c_{ijkl} has four subscripts, there are $3^4 = 81$ constants. However, stress symmetry ($\sigma_{ij} = \sigma_{ji}$) implies

$$c_{ijkl} = c_{jikl}, \tag{4.8}$$

and strain symmetry ($\varepsilon_{ij} = \varepsilon_{ji}$) implies

$$c_{ijkl} = c_{ijlk}. \tag{4.9}$$

This last result is proved by observing that

$$c_{ijkl}\varepsilon_{kl} = c_{ijlk}\varepsilon_{lk} = c_{ijlk}\varepsilon_{kl}, \tag{4.10}$$

where the first equation follows from the interchange of the dummy indices k and l, and the second equation follows from the strain symmetry $\varepsilon_{kl} = \varepsilon_{lk}$. Thus, since the constants c_{ijkl} relate six unique stresses to six unique strains, there are (so far) at most 36 unique constants. An alternative way to display the generalized Hooke's law is

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & c_{1123} & c_{1131} \\ c_{2211} & c_{2222} & c_{2233} & c_{2212} & c_{2223} & c_{2231} \\ c_{3311} & c_{3322} & c_{3333} & c_{3312} & c_{3323} & c_{3331} \\ c_{1211} & c_{1222} & c_{1233} & c_{1212} & c_{1223} & c_{1231} \\ c_{2311} & c_{2322} & c_{2333} & c_{2312} & c_{2323} & c_{2331} \\ c_{3111} & c_{3122} & c_{3133} & c_{3112} & c_{3123} & c_{3131} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{Bmatrix}, \tag{4.11}$$

where, for $i \neq j$, $2\varepsilon_{ij} = \gamma_{ij}$ is the engineering shear strain. Note that, although we have arranged the stresses and strains in Eq. 4.11 into arrays, these arrays are not vectors (tensors of rank 1). In computational mechanics, it is sometimes convenient to use arrays of stress and strain components like these.

To avoid dealing with the double sums in generalized Hooke's law, we could alternatively introduce the notation

$$\begin{aligned} &\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{33} = \sigma_3, \quad \sigma_{12} = \sigma_4, \quad \sigma_{23} = \sigma_5, \quad \sigma_{31} = \sigma_6, \\ &\varepsilon_{11} = \varepsilon_1, \quad \varepsilon_{22} = \varepsilon_2, \quad \varepsilon_{33} = \varepsilon_3, \quad 2\varepsilon_{12} = \varepsilon_4, \quad 2\varepsilon_{23} = \varepsilon_5, \quad 2\varepsilon_{31} = \varepsilon_6, \end{aligned} \tag{4.12}$$

in which case generalized Hooke's law could be written in the form

$$\sigma_i = c_{ij}\varepsilon_j. \tag{4.13}$$

4.2 Strain Energy

We consider first the one-dimensional case of a rod of length L placed in uniaxial tension by a force F (Fig. 22). We let u denote the axial displacement of the rod (the change in length of the rod). Let F be increased gradually from zero to some level $\bar{F}$. By Hooke's law, the force *vs.* displacement curve is linear, as shown in Fig. 23:

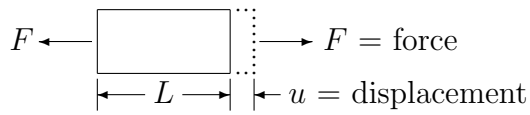


Figure 22: Rod in Uniaxial Tension.

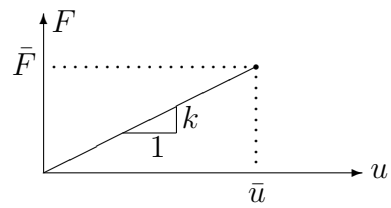


Figure 23: Force-Displacement Curve for Rod in Uniaxial Tension.

$$F = ku, \tag{4.14}$$

where k is the stiffness (force per unit length) of the rod. The work W done by the force on the rod is the area under the force-displacement curve:

$$W = \int_0^{\bar{u}} F(u)\,du = \int_0^{\bar{u}} ku\,du = \frac{1}{2}k\bar{u}^2 = \frac{1}{2}\bar{F}\bar{u} = \frac{1}{2}\sigma A\varepsilon L = \frac{1}{2}\sigma\varepsilon V, \tag{4.15}$$

where $\bar{u}$ is the final displacement corresponding to force $\bar{F}$, A is the cross-sectional area of the rod, and $V = AL$ is the volume of the rod. The work done by the force F on the rod goes into increasing the internal energy (called *strain energy*) in the material.

Let w denote strain energy density (the strain energy per unit volume). For the one-dimensional rod case,

$$w = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}\sigma_{xx}\varepsilon_{xx}. \tag{4.16}$$

We note that, for this one-dimensional case,

$$\frac{\partial w}{\partial \varepsilon_{xx}} = \frac{\partial}{\partial \varepsilon_{xx}}\left(\frac{1}{2}E\varepsilon_{xx}^2\right) = E\varepsilon_{xx} = \sigma_{xx}. \tag{4.17}$$

That is, the derivative of strain energy density with respect to strain yields the stress.

For multiaxial stress, we define the strain energy density as

$$w = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{4.18}$$

or, if we substitute generalized Hooke's law (Eq. 4.2),

$$w = \frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}. \tag{4.19}$$

In the more compact notation of Eq. 4.13, Eq. 4.19 could alternatively be written

$$w = \frac{1}{2}c_{ij}\varepsilon_i\varepsilon_j, \tag{4.20}$$

which is a quadratic form. Since, from the discussion of quadratic forms in §1.3, the anti-symmetric part of c_{ij} does not contribute to w, we could take

$$c_{ij} = c_{ji}, \tag{4.21}$$

or, in terms of the material tensor of rank 4,

$$c_{ijkl} = c_{klij}. \tag{4.22}$$

If we differentiate the strain energy density function with respect to a strain component, we obtain

$$\begin{aligned}\frac{\partial w}{\partial \varepsilon_{mn}} &= \frac{1}{2}c_{ijkl}\frac{\partial \varepsilon_{ij}}{\partial \varepsilon_{mn}}\varepsilon_{kl} + \frac{1}{2}c_{ijkl}\varepsilon_{ij}\frac{\partial \varepsilon_{kl}}{\partial \varepsilon_{mn}} = \frac{1}{2}c_{ijkl}\delta_{im}\delta_{jn}\varepsilon_{kl} + \frac{1}{2}c_{ijkl}\varepsilon_{ij}\delta_{km}\delta_{ln} \\ &= \frac{1}{2}c_{mnkl}\varepsilon_{kl} + \frac{1}{2}c_{ijmn}\varepsilon_{ij} = \frac{1}{2}c_{mnij}\varepsilon_{ij} + \frac{1}{2}c_{mnij}\varepsilon_{ij} = c_{mnij}\varepsilon_{ij} = \sigma_{mn},\end{aligned} \tag{4.23}$$

where, in the fourth equation, we replaced the dummy subscripts kl with ij in the first term and used the symmetry given in Eq. 4.22 in the second term. Thus, with this assumed symmetry, the derivative of the strain energy density with respect to a component of strain is the corresponding component of stress. We can differentiate again to obtain

$$\frac{\partial^2 w}{\partial \varepsilon_{kl}\partial \varepsilon_{mn}} = c_{mnij}\frac{\partial \varepsilon_{ij}}{\partial \varepsilon_{kl}} = c_{mnij}\delta_{ik}\delta_{jl} = c_{mnkl}. \tag{4.24}$$

Similarly,

$$\frac{\partial^2 w}{\partial \varepsilon_{mn}\partial \varepsilon_{kl}} = c_{klmn}. \tag{4.25}$$

Since the left-hand sides of the last two equations are equal, we again obtain the result

$$c_{ijkl} = c_{klij}. \tag{4.26}$$

To avoid the appearance of a circular argument here, we summarize the basic result: If there exists a strain energy density function w given by Eq. 4.19 with the property

$$\frac{\partial w}{\partial \varepsilon_{ij}} = \sigma_{ij}, \tag{4.27}$$

Eq. 4.26 holds.

Since we started with strain energy, this symmetry of the material constants is implied by the existence of strain energy. As a consequence of Eq. 4.26, the 6×6 matrix of material constants in Eq. 4.11 is symmetric. Thus, the existence of a strain energy function implies that there are at most 21 independent material constants. That is, the most general anisotropic homogeneous material has at most 21 elastic constants.

4.3 Material Symmetry

We have shown that the most general anisotropic homogeneous material has at most 21 independent elastic constants. However, a particular material may exhibit certain symmetries with regard to material properties. Such symmetries reduce further the number of independent elastic constants.

For example, a material which is elastically symmetric with respect to the xy-plane would have elastic constants c_{ijkl} which are invariant under a coordinate transformation corresponding to a reflection through the xy-plane. That is, the components of a vector $\mathbf{x}$ would transform according to

$$\begin{cases} \bar{x}_1 = x_1, \\ \bar{x}_2 = x_2, \\ \bar{x}_3 = -x_3, \end{cases} \tag{4.28}$$

which corresponds to a coordinate transformation matrix given by

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{4.29}$$

or

$$R_{ij} = a_i \delta_{ij} \quad \text{(no sum on } i\text{)}, \tag{4.30}$$

where

$$\mathbf{a} = \begin{Bmatrix} 1 \\ 1 \\ -1 \end{Bmatrix}. \tag{4.31}$$

For this material,

$$c_{ijkl} = R_{im} R_{jn} R_{ko} R_{lp} c_{mnop}, \tag{4.32}$$

where we have c rather than $\bar{c}$ on the left-hand side, since the material constants are invariant under the symmetry transformation. Thus,

$$c_{ijkl} = a_i \delta_{im} a_j \delta_{jn} a_k \delta_{ko} a_l \delta_{lp} c_{mnop} = a_i a_j a_k a_l c_{ijkl} \quad \text{(no sum on } i, j, k, l\text{)}, \tag{4.33}$$

where $a_i = \pm 1$. This last result implies that, if $a_i a_j a_k a_l \neq 1$, $c_{ijkl} = 0$, which implies that, if any *odd* number of subscripts is equal to 3, $c_{ijkl} = 0$. Hence,

$$c_{3111} = c_{3112} = c_{3122} = c_{3211} = c_{3212} = c_{3222} = c_{3331} = c_{3332} = 0, \tag{4.34}$$

and the number of independent elastic constants reduces from 21 to 13:

$$\mathbf{C} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1112} & 0 & 0 \\ & c_{2222} & c_{2233} & c_{2212} & 0 & 0 \\ & & c_{3333} & c_{3312} & 0 & 0 \\ & & & c_{1212} & 0 & 0 \\ & \text{Sym} & & & c_{2323} & c_{2331} \\ & & & & & c_{3131} \end{bmatrix}, \tag{4.35}$$

where $\mathbf{C}$ denotes the matrix of elastic constants. This symmetry case is useful, because it leads to an easy derivation of the orthotropic case.

An *orthotropic* material has three orthogonal planes of symmetry. Thus, in Eq. 4.35, those elastic constants having an odd number of subscripts equal to either 1 or 2 will vanish, and the material matrix is

$$\mathbf{C} = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & 0 & 0 & 0 \\ & c_{2222} & c_{2233} & 0 & 0 & 0 \\ & & c_{3333} & 0 & 0 & 0 \\ & & & c_{1212} & 0 & 0 \\ & \text{Sym} & & & c_{2323} & 0 \\ & & & & & c_{3131} \end{bmatrix}. \tag{4.36}$$

Thus, an orthotropic material has nine independent elastic constants. Note that, for an orthotropic material, there is no coupling between the direct stresses and the shear strains and *vice versa*.

An *isotropic* material, for which all directions are equivalent, has two independent elastic constants. We will prove this statement using a tensor approach rather than using the above approach, which would require the application of various rotations [18].

4.4 Isotropic Materials

We recall from the end of §1.2 that the Kronecker delta δ_{ij} is an isotropic tensor of rank 2. It can be shown in tensor analysis that δ_{ij} is the only isotropic tensor of rank 2 and, moreover, δ_{ij} is the characteristic tensor for all isotropic tensors:

Rank	Isotropic Tensors
1	none
2	$c\delta_{ij}$
3	none
4	$a\delta_{ij}\delta_{kl} + b\delta_{ik}\delta_{jl} + c\delta_{il}\delta_{jk}$
odd	none

(The alternating symbol e_{ijk} is a pseudotensor.) All isotropic tensors of rank 4 must be of the form shown above. Specifically, for the tensor c_{ijkl} to be an isotropic tensor of rank 4, it must necessarily be of the form

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \beta\delta_{il}\delta_{jk}, \tag{4.37}$$

where the symmetry $c_{ijkl} = c_{jikl}$ implies $\beta = \mu$. Thus, for an isotropic material,

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{4.38}$$

where λ and μ are referred to as the Lamé constants of elasticity. Thus, an isotropic material has two independent elastic constants.

Using Eq. 4.38, the generalized Hooke's law becomes

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} = \lambda\delta_{ij}\delta_{kl}\varepsilon_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\varepsilon_{kl} = \lambda\varepsilon_{kk}\delta_{ij} + \mu\varepsilon_{ij} + \mu\varepsilon_{ji}. \tag{4.39}$$

Thus, because the strain tensor in symmetric, generalized Hooke's law becomes, for an isotropic material,

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{4.40}$$

which has two independent elastic constants. In expanded form, we obtain

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{Bmatrix}. \tag{4.41}$$

No further reduction in the number of elastic constants is possible. The Lamé constants λ and μ will be related to the more familiar engineering constants (Young's modulus E, Poisson's ratio ν, and the shear modulus G) in the next chapter.

To determine the strain-stress equations (the inverse of the stress-strain equations) for an isotropic material, we first use Eq. 4.40 to obtain

$$\sigma_{ii} = (3\lambda + 2\mu)\varepsilon_{ii}, \tag{4.42}$$

since $\delta_{ii} = 3$ (the trace of the identity matrix). Then,

$$2\mu\varepsilon_{ij} = \sigma_{ij} - \lambda\delta_{ij}\left(\frac{\sigma_{kk}}{3\lambda + 2\mu}\right) \tag{4.43}$$

or

$$\varepsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right]. \tag{4.44}$$

5 Simplest Problems of Elastostatics

The momentum equation, Eq. 3.26a, reduces for statics problems to the equilibrium equation

$$\sigma_{ij,j} = -\rho f_i, \tag{5.1}$$

since the acceleration term is zero. In the absence of body forces (e.g., gravity), the equilibrium equation is simply

$$\sigma_{ij,j} = 0. \tag{5.2}$$

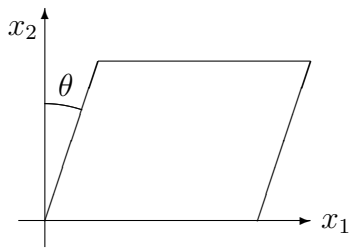


Figure 24: Simple Shear.

5.1 Simple Shear

Consider the simple shearing deformation

$$\begin{cases} u_1 = kx_2, \\ u_2 = 0, \\ u_3 = 0, \end{cases} \tag{5.3}$$

where $k = \tan\theta$ is an infinitesimal constant (Fig. 24). For this deformation, the displacement gradient matrix $u_{i,j}$ is given by

$$(u_{i,j}) = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5.4}$$

and the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ (which is the symmetric part of this matrix) is, from Eq. 2.65,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.5}$$

For an isotropic material, we compute the stresses from the generalized Hooke's law, Eq. 4.40, where, for this shearing deformation, $\varepsilon_{kk} = 0$. Thus, the stress field is given by

$$\begin{cases} \sigma_{12} = \sigma_{21} = \mu k, \\ \sigma_{ij} = 0 \text{ otherwise.} \end{cases} \tag{5.6}$$

That is, there are only two nonzero components of stress. Since $k = 2\varepsilon_{12} = \gamma_{12}$ is the engineeering shear strain, $\sigma_{12} = \mu\gamma_{12}$, and the Lamé constant μ is the shear modulus (usually called G in engineering). Note that, since $\varepsilon_{ii} = 0$, there is no volume change for simple shear.

5.2 Simple Tension

Consider a uniform rod of length L and cross-sectional area A in simple tension, as shown in Fig. 25. If the applied force F is applied uniformly over each end face, $\sigma_{xx} = \sigma = F/A$,

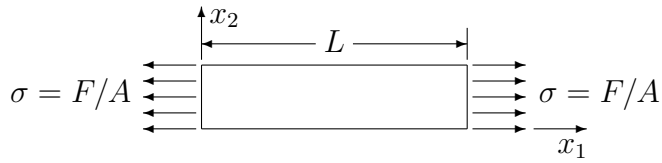


Figure 25: Simple Tension.

and the stress tensor $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.7}$$

The strains are computed from the inverse form of Hooke's law (Eq. 4.44) for an isotropic material, where $\sigma_{kk} = \sigma$, and $\varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = 0$. That is, for an isotropic material, there are no shear strains for this problem. The component of strain in the direction of the load is

$$\varepsilon_{11} = \frac{1}{2\mu}\left[\sigma - \frac{\lambda\sigma}{3\lambda + 2\mu}\right] = \frac{\sigma}{\mu} \cdot \frac{\lambda + \mu}{3\lambda + 2\mu}. \tag{5.8}$$

The ratio of axial stress to axial strain for simple tension is defined as the *Young's modulus* E for the material:

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \tag{5.9}$$

E is also known as the *modulus of elasticity*. Thus, $\varepsilon_{11} = \sigma/E$.

From Eq. 4.44, we can also obtain the normal strains orthogonal to the direction of load:

$$\varepsilon_{22} = \varepsilon_{33} = -\frac{1}{2\mu} \cdot \frac{\lambda}{3\lambda + 2\mu}\,\sigma = -\frac{\lambda}{2(\lambda + \mu)}\varepsilon_{11}, \tag{5.10}$$

where Eq. 5.8 was substituted. We now define *Poisson's ratio* ν as the negative of the ratio of transverse strain to normal strain for the simple tension problem. Thus, with $\nu = -\varepsilon_{22}/\varepsilon_{11}$,

$$\nu = \frac{\lambda}{2(\lambda + \mu)}. \tag{5.11}$$

The negative sign is used in the definition of Poisson's ratio since, for most materials, a stretch in the loaded direction causes shrinkage in the transverse direction. Some materials (e.g., cork) have a Poisson's ratio near zero. A few materials have been developed with negative Poisson's ratio.

Note that, from Eq. 5.9, E can be written in the form

$$E = \mu\frac{\lambda + 2(\lambda + \mu)}{\lambda + \mu} = \mu\left(\frac{\lambda}{\lambda + \mu} + 2\right) = \mu(2\nu + 2) = 2\mu(1 + \nu). \tag{5.12}$$

Thus, the relationship among the three elastic constants for an isotropic material is

$$G = \mu = \frac{E}{2(1 + \nu)}, \tag{5.13}$$

where G is the symbol commonly used in engineering for the shear modulus. This equation can also be viewed as the equation relating the Lamé constant μ to the engineering constants E and ν.

To solve for the second Lamé constant in terms of the engineering constants, we solve Eq. 5.11 for λ to obtain

$$\lambda = \frac{2\nu\mu}{1-2\nu} = \frac{E}{1+\nu} \cdot \frac{\nu}{1-2\nu} = \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{5.14}$$

To summarize, the Lamé constants are related to the engineering constants E (Young's modulus) and ν (Poisson's ratio) by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{5.15}$$

or, inversely,

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}. \tag{5.16}$$

The Lamé constant μ is the shear modulus G.

Using these relationships, we can also write the generalized Hooke's law in terms of the engineering constants instead of the Lamé constants. If we substitute Eqs. 5.15 into Hooke's law, Eq. 4.40, we obtain

$$\sigma_{ij} = \frac{E}{1+\nu}\left[\frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij} + \varepsilon_{ij}\right]. \tag{5.17}$$

The inverse relationship (strain-stress) is obtained from Eq. 4.44, where

$$3\lambda + 2\mu = \frac{3E\nu}{(1+\nu)(1-2\nu)} + \frac{E}{1+\nu} = \frac{E(3\nu+1-2\nu)}{(1+\nu)(1-2\nu)} = \frac{E}{1-2\nu}, \tag{5.18}$$

and

$$\frac{\lambda}{3\lambda+2\mu} = \frac{\nu}{1+\nu}. \tag{5.19}$$

Thus, from Eq. 4.44,

$$\varepsilon_{ij} = \frac{1+\nu}{E}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right] \tag{5.20}$$

or

$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}]. \tag{5.21}$$

Eq. 5.21 can be expanded to yield, for example,

$$\varepsilon_{11} = \frac{1}{E}[(1+\nu)\sigma_{11} - \nu(\sigma_{11}+\sigma_{22}+\sigma_{33})] = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}. \tag{5.22}$$

Similarly, we can expand the other two normal strains to obtain

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}, \tag{5.23}$$

$$\varepsilon_{22} = -\nu\frac{\sigma_{11}}{E} + \frac{\sigma_{22}}{E} - \nu\frac{\sigma_{33}}{E}, \tag{5.24}$$

$$\varepsilon_{33} = -\nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}, \tag{5.25}$$

which is the form of the strain-stress equations we would expect, given the definitions of E and ν.

It can be shown[18] that, by substituting inverse Hooke's law for an isotropic material (Eq. 5.21) into the compatibility equations for strain (Eq. 2.86), and making use of the equilibrium equations with zero body force ($\sigma_{ij,j} = 0$), we would obtain the Beltrami compatibility equations in terms of stress:

$$(1+\nu)\nabla^2\sigma_{ij} + \sigma_{kk,ij} = 0 \quad (i,j = 1,2,3). \tag{5.26}$$

A more general form of these equations, known as the Beltrami-Michell equations (not shown here), results if body forces are included in the equilibrium equations. Thus, if stresses are proposed as the solution of an elasticity problem, those stresses must satisfy both equilibrium and the compatibility equations, where compatibility can be checked either directly using the Beltrami equations or indirectly by first using inverse Hooke's law to convert stresses to strains and then using the strain compatibility equations.

5.3 Uniform Compression

Consider the uniform displacement field $u_i = \varepsilon x_i$ for a body, in which case the displacement gradient matrix is

$$(u_{i,j}) = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix} = \boldsymbol{\varepsilon} = \varepsilon\mathbf{I}. \tag{5.27}$$

The dilatation, denoted Δ, is the volumetric strain $\Delta V/V$ given by

$$\Delta = \frac{\Delta V}{V} = u_{i,i} = \text{tr } \boldsymbol{\varepsilon} = 3\varepsilon. \tag{5.28}$$

From Hooke's law for an isotropic material, Eq. 4.40,

$$\sigma_{ij} = 3\varepsilon\lambda\delta_{ij} + 2\mu\varepsilon\delta_{ij} = 3\varepsilon\left(\lambda + \frac{2}{3}\mu\right)\delta_{ij} = \Delta\left(\lambda + \frac{2}{3}\mu\right)\delta_{ij}. \tag{5.29}$$

If $-p$ denotes the *mean normal stress*,

$$-p = \frac{1}{3}\sigma_{ii}, \tag{5.30}$$

then, for this problem,

$$-p = \sigma_{11} = \sigma_{22} = \sigma_{33} = \left(\lambda + \frac{2}{3}\mu\right)\Delta. \tag{5.31}$$

We now define the *bulk modulus* (or *modulus of compression*) K as

$$K = \frac{p}{-\Delta} = \frac{p}{-(\Delta V/V)} = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)}, \tag{5.32}$$

where the last equation follows from Eq. 5.18.

Note that $\nu > \frac{1}{2}$ implies $K < 0$. That is, a uniform positive pressure applied to a body would result in an *increase* in volume, which is not possible (proved in §5.5). Hence, for physically meaningful deformations,

$$\nu \leq \frac{1}{2}. \tag{5.33}$$

The special case $\nu = 1/2$ implies an infinite bulk modulus, which means the material is incompressible. Rubber is a material which is nearly incompressible.

From Eq. 5.29, the stress tensor for uniform compression can alternatively be expressed as

$$\sigma_{ij} = \left(\lambda + \frac{2}{3}\mu\right)\Delta\delta_{ij} = K\varepsilon_{kk}\delta_{ij}. \tag{5.34}$$

Bulk moduli for a few common materials are shown in the following table:

Material	Bulk modulus K
water	2.2 GPa
rubber	2.5
aluminum	70
steel	160
diamond	620

The criterion for a solid material to be considered incompressible is that the ratio K/G is large for the material, not having a large K relative to that of other materials. Thus, water is sometimes treated as being incompressible, but steel never is. Rubber is generally considered incompressible. It can be shown that, for water, $K = \rho c^2$, where ρ is the density, and c is the speed of sound. For isotropic solid materials, K is related to the other elastic constants by Eq. 5.32.

5.4 Stress and Strain Deviators

Earlier in this chapter, we considered simple shear, which involved a shape change but no volume change, and uniform compression, which involved a volume change but no shape change. In simple shear, the stress field is given by $\sigma_{ij} = 2\mu\varepsilon_{ij}$, where $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0$ and $\varepsilon_{ii} = u_{i,i} = 0$ (no volume change). In uniform compression, $\sigma_{ij} = K\varepsilon_{kk}\delta_{ij}$, where $\boldsymbol{\varepsilon} = \varepsilon\mathbf{I}$ (no change in shape). The uniform compression problem is sometimes called *hydrostatic compression* or *isotropic compression*.

We wish to show here that any general deformation can be represented as the sum of a pure shear and a hydrostatic compression. For a general deformation, the strain can be written in the form

$$\varepsilon_{ij} = \left(\varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right) + \frac{1}{3}\varepsilon_{kk}\delta_{ij}, \tag{5.35}$$

where the last term has been both subtracted and added. However, the expression in parentheses represents a simple shear, because its trace (the sum of the diagonal terms) is zero. The second term is a hydrostatic compression.

If we define the *deviatoric strain* (or *strain deviator*) as

$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}, \tag{5.36}$$

then any general strain can be written as

$$\varepsilon_{ij} = e_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij}, \tag{5.37}$$

where the first term is the deviatoric strain, and the second term is the isotropic strain. Note that the trace e_{ii} of the deviatoric strain vanishes, as required for pure shear.

Similarly, any stress field can be represented as the sum of pure shear and hydrostatic compression components:

$$\sigma_{ij} = s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij} = s_{ij} - p\delta_{ij}, \tag{5.38}$$

where the first term is the *deviatoric stress* (or *stress deviator*) defined as

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}, \tag{5.39}$$

and the second term ($-p\delta_{ij}$) is the isotropic part of the stress, with p the pressure. Note that the trace s_{ii} of the deviatoric stress vanishes.

We can use Hooke's law for an isotropic material, Eq. 4.40, to relate the deviatoric and isotropic parts of stress and strain to each other:

$$s_{ij} - p\delta_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\left(e_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right). \tag{5.40}$$

Since the traces of the deviatoric stress and strain tensors vanish, we can contract this equation (i.e., set $i = j$) to obtain

$$-3p = 3\lambda\varepsilon_{kk} + 2\mu\varepsilon_{kk} = (3\lambda + 2\mu)\varepsilon_{kk} \tag{5.41}$$

or

$$-p = \left(\lambda + \frac{2}{3}\mu\right)\varepsilon_{kk} = K\varepsilon_{kk}. \tag{5.42}$$

This last result establishes the relationship between the isotropic parts of stress and strain.

To determine the relationship between the deviatoric parts of stress and strain, we substitute Eq. 5.42 into Eq. 5.40 to obtain

$$s_{ij} + \left(\lambda + \frac{2}{3}\mu\right)\varepsilon_{kk}\delta_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\left(e_{ij} + \frac{1}{3}\varepsilon_{kk}\delta_{ij}\right), \tag{5.43}$$

where, after cancellation, we are left with

$$s_{ij} = 2\mu e_{ij}. \tag{5.44}$$

Thus, for an isotropic material, Hooke's law, Eq. 4.40, can be written in the equivalent form

$$\begin{cases} s_{ij} = 2\mu e_{ij} \\ \frac{1}{3}\sigma_{ii} = K\varepsilon_{kk}, \end{cases} \tag{5.45}$$

where s_{ij} and e_{ij} are the deviatoric (pure shear) components of stress and strain, respectively.

5.5 Stable Reference States

We recall the strain energy density defined in Eq. 4.18 as

$$w = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}. \tag{5.46}$$

We can write w in terms of the deviatoric and isotropic components as

$$w = \frac{1}{2}\left(s_{ij} + \frac{1}{3}\sigma_{kk}\delta_{ij}\right)\left(e_{ij} + \frac{1}{3}\varepsilon_{ll}\delta_{ij}\right) \tag{5.47}$$

$$= \frac{1}{2}\left(s_{ij}e_{ij} + \frac{1}{3}\sigma_{kk}e_{ii} + \frac{1}{3}\varepsilon_{kk}s_{ii} + \frac{1}{3}\sigma_{kk}\varepsilon_{ll}\right), \tag{5.48}$$

where $\delta_{ij}\delta_{ij} = \delta_{ii} = 3$. The second and third terms of this expression vanish, since the trace e_{ii} of deviatoric strain and trace s_{ii} of deviatoric stress both vanish. Thus,

$$w = \frac{1}{2}s_{ij}e_{ij} + \frac{1}{2}\left(\frac{1}{3}\sigma_{kk}\right)\varepsilon_{ll}. \tag{5.49}$$

This result is applicable for anisotropic materials. For an isotropic material, Eq. 5.45 implies

$$w = \mu e_{ij}e_{ij} + \frac{1}{2}K(\varepsilon_{ii})^2, \tag{5.50}$$

where the contributions to w from deviatoric strain and isotropic strain uncouple.

We now define a reference state as *stable* if the strain energy density $w > 0$ for all $\boldsymbol{\varepsilon} \neq \mathbf{0}$. That is, the work required to strain a body is positive. Thus, for a stable reference state,

$$2w = c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > 0 \quad (\boldsymbol{\varepsilon} \neq \mathbf{0}), \tag{5.51}$$

or c_{ijkl} is positive definite (where we have generalized the definition of positive definiteness given at the end of §1.3 for tensors of rank 2).

Consider, in particular, an isotropic material, where the strain energy density is given by Eq. 5.50. For a pure expansion, for which the deviatoric strain $e_{ij} = 0$, $w > 0$ implies $K > 0$. Also, for pure shear, for which the volumetric strain $\varepsilon_{ii} = 0$, $w > 0$ implies $\mu > 0$. Thus, a necessary and sufficient condition for a stable reference state for an isotropic material is

$$K > 0, \quad \mu > 0. \tag{5.52}$$

Positive values for these two material constants also imply restrictions on E and ν, as shown in the table below:

$2\mu = E/(1+\nu) > 0$ (Eq. 5.13)	$3K = E/(1-2\nu) > 0$ (Eq. 5.32)
$E > 0$ and $\nu > -1$	$E > 0$ and $\nu < \frac{1}{2}$
or $E < 0$ and $\nu < -1$	or $E < 0$ and $\nu > \frac{1}{2}$

Since the two sides of the second row above are inconsistent, only the first row is possible. Thus, for stability for isotropic materials,

$$E > 0, \quad -1 < \nu < \frac{1}{2}. \tag{5.53}$$

6 Boundary Value Problems in Elastostatics

We start by summarizing the basic equations of static elasticity:

$\sigma_{ij,j} + \rho f_i = 0$	(equilibrium equations)
$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$	(Hooke's law)
$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$	(strain-displacement equations)
$t_i = \sigma_{ij} n_j$	(traction boundary conditions)

Since Hooke's law can be written in the form

$$\sigma_{ij} = c_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}) = c_{ijkl}u_{k,l}, \tag{6.1}$$

the equilibrium equations can be written as

$$(c_{ijkl}u_{k,l})_{,j} + \rho f_i = 0 \quad (i = 1, 2, 3). \tag{6.2}$$

This coupled system of three partial differential equations is called the *Navier equations of elasticity*. The unknowns in this system are the three Cartesian components of displacement. These equations are applicable for anisotropic materials.

For isotropic materials, Hooke's law is given by Eq. 4.40:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}). \tag{6.3}$$

We differentiate to obtain

$$\sigma_{ij,j} = \lambda u_{k,kj}\delta_{ij} + \mu(u_{i,jj} + u_{j,ij}) = \lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) = \mu u_{i,jj} + (\lambda + \mu)u_{j,ji}, \tag{6.4}$$

where we have assumed that the material is also homogeneous (since, otherwise, the material constants λ and μ would be position-dependent and would have to be differentiated). A *homogeneous material* is one whose properties are independent of position. Thus, the Navier equations for an isotropic homogeneous material are

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \rho f_i = 0 \quad (i = 1, 2, 3). \tag{6.5}$$

The goal of a boundary value problem in elasticity is to determine the distribution of stress and the displacements in the interior of an elastic body in equilibrium when there

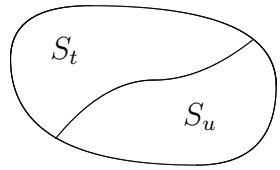


Figure 26: Surfaces for Boundary Value Problem.

are specified (a) boundary displacements, (b) boundary tractions, or (c) a combination of boundary displacements and tractions.

Consider the surface S of a body consisting of a part S_u on which displacements u_i are specified and a part S_t on which tractions $t_i = \sigma_{ij} n_j$ are specified (Fig. 26). In a displacement problem, $S = S_u$, and S_t is empty. In a traction problem, $S = S_t$, and S_u is empty. In a mixed problem, $S = S_u \cup S_t$, where $S_u \cap S_t = \emptyset$. S_u and S_t need not be contiguous.

It turns out that the displacement and mixed problems are well-posed (i.e., the solution exists and is unique), but the traction problem is not well-posed. The non-uniqueness which exists for the traction problem is that a rigid body rotation and displacement can be superposed on any displacement solution to get another solution. Since stresses are determined from displacement gradients, the stress solution in a traction problem is unique.

6.1 Uniqueness

In a displacement boundary value problem, it is desired to find the vector field $\mathbf{u}$ satisfying the system

$$\begin{cases} (c_{ijkl} u_{k,l})_{,j} = -\rho f_i & \text{in } V, \\ u_i = U_i & \text{on } S, \end{cases} \tag{6.6}$$

where S is the surface of the closed volume V, and U_i is the vector of prescribed displacements on the boundary. That is, this system of partial differential equations must hold at all points interior to the volume V, and the displacement boundary condition must hold at all points on the surface S.

We define a boundary value problem as *homogeneous* if every term of both the partial differential equation and the boundary condition is proportional to the unknown solution u_i. Otherwise the boundary value problem is *nonhomogeneous*. The above system, Eq. 6.6, is nonhomogeneous. This use of the word "homogeneous" has nothing to do with its use in describing material properties.

We seek to prove that Eq. 6.6 has a unique solution. Assume that the solution is not unique, and there is another solution v_i. The second solution $\mathbf{v}$ would also satisfy Eq. 6.6:

$$\begin{cases} (c_{ijkl} v_{k,l})_{,j} = -\rho f_i & \text{in } V, \\ v_i = U_i & \text{on } S. \end{cases} \tag{6.7}$$

If we subtract the boundary value problems for $\mathbf{u}$ and $\mathbf{v}$, we obtain the boundary value problem which must be satisfied by the difference solution $\hat{w}_i = u_i - v_i$:

$$\begin{cases} (c_{ijkl} \hat{w}_{k,l})_{,j} = 0 & \text{in } V, \\ \hat{w}_i = 0 & \text{on } S. \end{cases} \tag{6.8}$$

Notice that this problem is the homogeneous problem corresponding to the original nonhomogeneous problem, Eq. 6.6. That is, the two nonhomogeneous terms $-\rho f_i$ and U_i are missing from Eq. 6.8. This problem, like all homogeneous problems, is satisfied by the trivial solution $\hat{w}_i = 0$. If $\hat{w}_i = 0$ is the *only* solution of Eq. 6.8, the original nonhomogeneous problem would have a unique solution. Thus, proving that a nonhomogeneous linear boundary value problem has a unique solution is equivalent to proving that the corresponding homogeneous problem has a unique solution.

That is, to prove uniqueness for the original nonhomogeneous problem, it suffices to show that $u_i = 0$ is the only solution of

$$\begin{cases} (c_{ijkl}u_{k,l})_{,j} = 0 & \text{in } V, \\ u_i = 0 & \text{on } S. \end{cases} \tag{6.9}$$

Eq. 6.9a implies that

$$0 = \int_V u_i(c_{ijkl}u_{k,l})_{,j}\, dV = \int_V [(u_i c_{ijkl}u_{k,l})_{,j} - c_{ijkl}u_{i,j}u_{k,l}]\, dV \tag{6.10}$$

$$= \oint_S u_i c_{ijkl}u_{k,l}n_j\, dS - \int_V c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}\, dV \tag{6.11}$$

$$= \oint_S u_i\sigma_{ij}n_j\, dS - \int_V 2w\, dV \tag{6.12}$$

$$= \oint_S u_i t_i\, dS - \int_V 2w\, dV = -\int_V 2w\, dV, \tag{6.13}$$

where Eq. 6.11 follows from the divergence theorem, and the last equation follows since $u_i = 0$ on S. (This conversion above of a single volume integral into the difference of a surface integral and another volume integral is the three-dimensional equivalent of integration by parts. The surface integral is the "boundary" term.) Hence,

$$\int_V w\, dV = 0. \tag{6.14}$$

However, for a stable reference state, $w > 0$ for $\boldsymbol{\varepsilon} \neq \mathbf{0}$, so that

$$\int_V w\, dV > 0 \quad \text{for } \boldsymbol{\varepsilon} \neq \mathbf{0}. \tag{6.15}$$

Thus, Eq. 6.14 implies $\boldsymbol{\varepsilon} \equiv 0$ or $\varepsilon_{ij} \equiv 0$ at all points in V. However, zero strain implies a deformation consisting only of a rigid body rotation and translation:

$$u_i = \omega_{ij}x_j + c_i, \tag{6.16}$$

where ω_{ij} and c_i are constants. Since $u_i = 0$ everywhere on the boundary S, it follows that $\omega_{ij} = 0$ and $c_i = 0$. Hence, $u_i \equiv 0$ in V, and uniqueness is proved.

Note that, to prove uniqueness, we assumed that the strain energy density w was positive definite, but we did not assume that the material was either isotropic or homogeneous. That is, the elastic constants c_{ijkl} can be position-dependent.

6.2 Uniqueness for the Traction Problem

In a traction boundary value problem, it is desired to find the vector field $\mathbf{u}$ satisfying the system

$$\begin{cases} (c_{ijkl}u_{k,l})_{,j} = -\rho f_i & \text{in } V, \\ \sigma_{ij}n_j = t_i & \text{on } S, \end{cases} \tag{6.17}$$

where S is the surface of the closed volume V, and t_i is the vector of prescribed tractions on the boundary. That is, the partial differential equation must hold at all points interior to the volume V, and the traction boundary condition must hold at all points on the surface S.

As we saw in the preceding section, uniqueness for this problem is equivalent to having uniqueness for the corresponding homogeneous problem:

$$\begin{cases} (c_{ijkl}u_{k,l})_{,j} = 0 & \text{in } V, \\ \sigma_{ij}n_j = 0 & \text{on } S. \end{cases} \tag{6.18}$$

We proceed here in exactly the same way as in the preceding section. Eq. 6.18 implies that

$$\int_V u_i(c_{ijkl}u_{k,l})_{,j}\, dV = 0, \tag{6.19}$$

which leads to

$$\int_V w\, dV = 0, \tag{6.20}$$

where w is the strain energy density. Since w is positive definite, $w \equiv 0$, and $\boldsymbol{\varepsilon} \equiv 0$. Hence, the displacement field must be, at most, a rigid body rotation and translation:

$$u_i = \omega_{ij}x_j + c_i, \tag{6.21}$$

where ω_{ij} and c_i are constants. In the displacement problem, the requirement that u_i vanish on the boundary implied $\omega_{ij} = 0$ and $c_i = 0$. However, in the traction problem, there is no such requirement. Indeed, in general, $\omega_{ij} \neq 0$ and $c_i \neq 0$. Thus, the displacements are not unique; there can be an arbitrary rigid body motion and still satisfy all traction boundary conditions. However, since the strain tensor $\varepsilon_{ij} \equiv 0$ (for the homogeneous problem), the strains and stresses are unique.

6.3 Uniqueness for the Mixed Problem

In the mixed problem, the displacements are specified on part of the boundary, and the tractions are specified on the rest of the boundary. At every point of the boundary, either the displacements or the tractions must be specified. In fact, since $\mathbf{u}$ and $\mathbf{t}$ are both vectors, it is also possible that, at a given point on the boundary, some components of displacement are specified, and the other components of traction are specified.

Uniqueness for the mixed problem follows immediately from Eq. 6.13, since, if either u_i or t_i vanishes on S, $w \equiv 0$. That is, in Eq. 6.13, the surface integral can be expanded to yield

$$\oint_S u_i t_i\, dS = \int_{S_u} u_i t_i\, dS + \int_{S_t} u_i t_i\, dS, \tag{6.22}$$

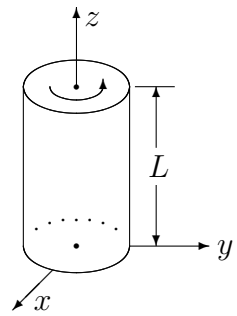


Figure 27: Torsion of Circular Shaft.

where $u_i = 0$ on S_u, and $t_i = 0$ on S_t. Moreover, since

$$u_i t_i = u_1 t_1 + u_2 t_2 + u_3 t_3, \tag{6.23}$$

the integrals could be further subdivided component by component.

To complete the uniqueness proof, $w \equiv 0$ implies $\boldsymbol{\varepsilon} \equiv 0$, and, hence, the displacements must be, at most, a rigid body rotation and translation:

$$u_i = \omega_{ij} x_j + c_i, \tag{6.24}$$

where ω_{ij} and c_i are constants. However, since the displacements must vanish on part of the boundary (S_u), the displacements must vanish identically throughout V, and uniqueness is proved.

7 Torsion

7.1 Circular Shaft

Consider a circular shaft of length L which is fixed at one end (the fixed base $z = 0$) and subjected at the other end ($z = L$) to a set of shear forces which are statically equivalent to a torque (Fig. 27). In general, "statically equivalent" means "same resultant force and moment." Rather than specify the tractions at $z = L$ precisely, we wish instead to propose a displacement field (solution), and see if all required equations are satisfied and if the resulting surface tractions and body forces are reasonable.

We assume that plane sections perpendicular to the z-axis remain plane. Let α denote the (infinitesimal) angle of twist per unit length. Consider a typical cross-section parallel to the xy-plane. A point P in that cross-section rotates slightly in the counter-clockwise direction by an infinitesimal amount θ (Fig. 28). The x and y components of the displacement of P are

$$\begin{aligned} u_1 &= -r\theta \sin\beta = (-r\theta)(y/r) = -\theta y, \\ u_2 &= r\theta \cos\beta = (r\theta)(x/r) = \theta x, \end{aligned} \tag{7.1}$$

where, for uniform twist, $\theta = \alpha z$.

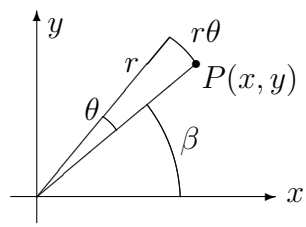


Figure 28: Geometry of Torsional Rotation.

Thus, the displacement field is

$$\begin{cases} u_1 = -\alpha yz, \\ u_2 = \alpha xz, \\ u_3 = 0, \end{cases} \tag{7.2}$$

from which it follows that the displacement gradient matrix is

$$(u_{i,j}) = \begin{bmatrix} 0 & -\alpha z & -\alpha y \\ \alpha z & 0 & \alpha x \\ 0 & 0 & 0 \end{bmatrix}, \tag{7.3}$$

and the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ (which is the symmetric part of this matrix) is, from Eq. 2.65,

$$\boldsymbol{\varepsilon} = \frac{1}{2}\begin{bmatrix} 0 & 0 & -\alpha y \\ 0 & 0 & \alpha x \\ -\alpha y & \alpha x & 0 \end{bmatrix} = \frac{\alpha}{2}\begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ -y & x & 0 \end{bmatrix}. \tag{7.4}$$

If we assume an isotropic material, Hooke's law, Eq. 4.40, applies:

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{7.5}$$

where, for this deformation, $\varepsilon_{kk} = 0$ (no volume change). Hence, the stress tensor is

$$\boldsymbol{\sigma} = \mu\alpha \begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ -y & x & 0 \end{bmatrix}. \tag{7.6}$$

In particular, the tractions on the top face of the shaft are given by the vector

$$\sigma_{3i} = \mu\alpha(-y, x, 0), \tag{7.7}$$

where we note that the vector $(-y, x, 0)$ is orthogonal to the vector $(x, y, 0)$. Since the latter vector is a radial vector, the vector of tractions on the top face is everywhere orthogonal to a radial vector; i.e., the tractions on the top face have rotational symmetry, a desirable characteristic for the tractions in a torsion problem (Fig. 29). In addition, this shear stress is proportional to the distance r from the center of the shaft. Thus, we could alternatively write the shear stress on the top face in the polar form

$$\sigma_{z\theta} = \mu\alpha r. \tag{7.8}$$

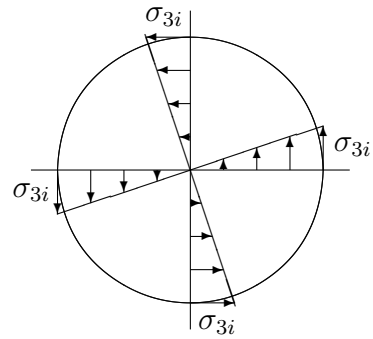


Figure 29: Rotational Symmetry of Shear Stress.

Another requirement for the tractions in a torsion problem is that they vanish on the side of the shaft. Since the unit normal $\mathbf{n}$ on the side is

$$\mathbf{n} = \left(\frac{x}{R}, \frac{y}{R}, 0\right), \tag{7.9}$$

where R is the radius of the shaft, the traction vector on the side is given by

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \frac{\mu\alpha}{R}\begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ -y & x & 0 \end{bmatrix}\begin{Bmatrix} x \\ y \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{7.10}$$

Thus, the surface tractions on the side vanish, as required.

We also must check that the equilibrium equations are satisfied:

$$\sigma_{ij,j} + \rho f_i = 0. \tag{7.11}$$

Here, $\sigma_{1i,i} = \sigma_{2i,i} = \sigma_{3i,i} = 0$. Hence, the equilibrium equations are satisfied, and the body forces also vanish. The strain compatiblity equations are automatically satisfied, since displacements, not strains, were originally proposed.

We therefore conclude that the original deformation can be supported by surface tractions alone, and we have a solution of the torsion problem for a circular shaft.

It is of interest to determine the relationship between the twisting moment (torque) and α, the angle of twist per unit length. The moment M at any cross-section can be obtained by integrating the moment of the shear stress $\sigma_{z\theta}$ over the cross-sectional area:

$$M = \int\int r\sigma_{z\theta}\, r\, dr\, d\theta, \tag{7.12}$$

where r is the moment arm, $dA = r\, dr\, d\theta$, and, from Eq. 7.8, $\sigma_{z\theta} = \mu\alpha r$. Consequently,

$$M = \int_0^{2\pi}\int_0^R r\mu\alpha r r\, dr\, d\theta = \mu\alpha \cdot \frac{R^4}{4} \cdot 2\pi = \mu\frac{\pi}{2}R^4\alpha. \tag{7.13}$$

This moment can also be expressed in terms of the polar moment of inertia for a circle,

$$I = \frac{\pi}{2}R^4, \tag{7.14}$$

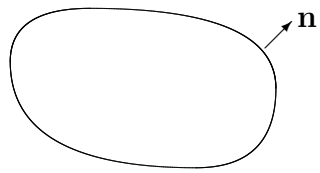


Figure 30: Torsion of Noncircular Shaft.

which we denote with I rather than J, since we want to reserve the symbol J for the torsional constant for non-circular shafts. For circular shafts, $J = I$. Thus,

$$M = \mu I \alpha = G I \alpha, \tag{7.15}$$

where $\mu = G$ is the shear modulus. We note that this solution is identical to that obtained in elementary mechanics of materials[15]:

$$\theta = \frac{ML}{IG}, \tag{7.16}$$

where M is the applied torque, and $\alpha = \theta/L$.

The method of solution for this problem is an example of the *semi-inverse method* of solution, in which the problem is attacked from several different sides simultaneously, with some assumptions concerning stresses and displacements, but with enough freedom to satisfy all the required conditions. Given an answer which satisfies all the equations, uniqueness implies that the solution derived is correct and unique.

7.2 Noncircular Shaft

We consider now the torsion of a shaft of noncircular cross-section. If we apply the same approach that was used for circular cross-sections, all the equations are unchanged up to the point where we computed the tractions on the side of the shaft. For the noncircular shaft, the unit outward normal $\mathbf{n}$ at a point on the boundary is the general normal (Fig. 30)

$$\mathbf{n} = (n_1, n_2, 0). \tag{7.17}$$

Thus, from Eq. 7.10, the traction vector on the side is given by

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \mu\alpha \begin{bmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ -y & x & 0 \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \\ 0 \end{Bmatrix} = \mu\alpha \begin{Bmatrix} 0 \\ 0 \\ -yn_1 + xn_2 \end{Bmatrix}. \tag{7.18}$$

where, in general, $t_3 \neq 0$. Moreover, at some points, $t_3 > 0$; at other points, $t_3 < 0$ (Fig. 31).

Since the requirement for zero tractions on the side of the shaft is not satisfied, we must start again to solve this problem. Our basic assumption was that plane sections remain plane. Perhaps this assumption needs to be relaxed, particularly in view of the fact that positive and negative tractions must be applied to the side to enforce this assumption.

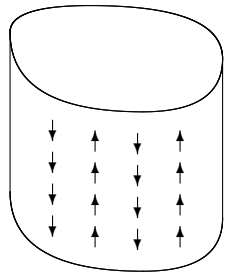


Figure 31: Tractions on Side of Noncircular Shaft.

Thus, similar to Eq. 7.2, we now propose the displacement field

$$\begin{cases} u_1 = -\alpha yz, \\ u_2 = \alpha xz, \\ u_3 = \alpha\phi(x,y), \end{cases} \tag{7.19}$$

where we now allow the cross-section to "warp" in the z direction, and $\phi(x,y)$ is called the *warping function*. This function will have to be determined as part of the problem solution. The displacement gradient matrix is then

$$(u_{i,j}) = \begin{bmatrix} 0 & -\alpha z & -\alpha y \\ \alpha z & 0 & \alpha x \\ \alpha\phi_{,x} & \alpha\phi_{,y} & 0 \end{bmatrix}, \tag{7.20}$$

where we use the notation

$$\phi_{,x} = \frac{\partial\phi}{\partial x}, \quad \phi_{,y} = \frac{\partial\phi}{\partial y}. \tag{7.21}$$

The infinitesimal strain tensor $\boldsymbol{\varepsilon}$ (which is the symmetric part of the displacement gradient matrix) is, from Eq. 2.65,

$$\boldsymbol{\varepsilon} = \frac{\alpha}{2}\begin{bmatrix} 0 & 0 & \phi_{,x} - y \\ 0 & 0 & \phi_{,y} + x \\ \phi_{,x} - y & \phi_{,y} + x & 0 \end{bmatrix}. \tag{7.22}$$

If we assume an isotropic material, Hooke's law, Eq. 4.40, applies:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \tag{7.23}$$

where, for this deformation, $\varepsilon_{kk} = 0$ (no volume change). Hence, the stress tensor is

$$\boldsymbol{\sigma} = \mu\alpha\begin{bmatrix} 0 & 0 & \phi_{,x} - y \\ 0 & 0 & \phi_{,y} + x \\ \phi_{,x} - y & \phi_{,y} + x & 0 \end{bmatrix}. \tag{7.24}$$

In particular, the tractions on the top face of the shaft, where the unit normal $\mathbf{n} = (0,0,1)$, are given by the vector

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \mu\alpha\begin{Bmatrix} \phi_{,x} - y \\ \phi_{,y} + x \\ 0 \end{Bmatrix}, \tag{7.25}$$

which lies in the xy-plane. On the side of the shaft, where the normal $\mathbf{n} = (n_1, n_2, 0)$, the vector of surface tractions is

$$\mathbf{t} = \boldsymbol{\sigma}\mathbf{n} = \mu\alpha \left\{ \begin{array}{c} 0 \\ 0 \\ (\phi_{,x} - y)n_1 + (\phi_{,y} + x)n_2 \end{array} \right\}. \tag{7.26}$$

Since we require $\mathbf{t} = \mathbf{0}$ on the side, we can view the equation $t_3 = 0$ as a boundary condition on ϕ. That is, we must choose ϕ in such a way that

$$\phi_{,x}n_1 + \phi_{,y}n_2 = n_1 y - n_2 x, \tag{7.27}$$

where the left-hand side is given by

$$\phi_{,x}n_1 + \phi_{,y}n_2 = \phi_{,i}n_i = \nabla\phi \cdot \mathbf{n} = \frac{\partial\phi}{\partial n}. \tag{7.28}$$

Thus, ϕ must satisfy the condition

$$\frac{\partial\phi}{\partial n} = n_1 y - n_2 x. \tag{7.29}$$

We also require that the equilibrium equations (with zero body force) be satisfied:

$$\sigma_{ij,j} = 0. \tag{7.30}$$

From Eq. 7.24,

$$\left\{ \begin{array}{l} \sigma_{1i,i} = 0, \\ \sigma_{2i,i} = 0, \\ \sigma_{3i,i} = \mu\alpha(\phi_{,xx} + \phi_{,yy}). \end{array} \right. \tag{7.31}$$

Thus, to satisfy equilibrium,

$$\phi_{,xx} + \phi_{,yy} = 0 \quad \text{or } \nabla^2\phi = 0. \tag{7.32}$$

That is, the warping function must satisfy the 2-D *Laplace equation*, solutions of which are said to be *harmonic* functions.

To summarize, solving the torsion problem for noncircular shafts reduces to solving the two-dimensional boundary value problem

$$\left\{ \begin{array}{l} \nabla^2\phi = 0 \quad \text{in } A, \\ \dfrac{\partial\phi}{\partial n} = n_1 y - n_2 x \quad \text{on } C, \end{array} \right. \tag{7.33}$$

as illustrated in Fig. 32. This problem is called a *Neumann* problem, since the boundary condition involves only the normal derivative of the unknown ϕ. As we will see in the next section, such problems have a unique solution only up to an arbitrary additive constant. (Partial differential equations whose boundary conditions specify the *value* of the unknown function rather than the *gradient* of the unknown function are known as *Dirichlet* problems.) Laplace's equation is an *elliptic* partial differential equation.

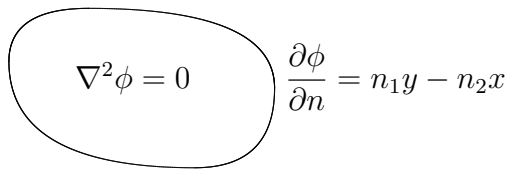


Figure 32: Warping Function Boundary Value Problem.

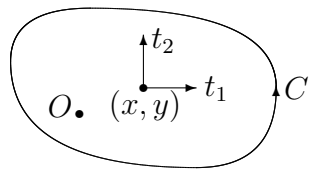


Figure 33: Calculation of End Moment in Torsion Problem.

We now compute the resultant force and moment on the top of the shaft. From Eq. 7.25, the resultant force in the x direction is

$$F_x = \mu\alpha \int_A (\phi_{,x} - y)\, dA = \mu\alpha \int_A \left\{ \frac{\partial}{\partial x}[x(-y + \phi_{,x})] + \frac{\partial}{\partial y}[x(x + \phi_{,y})] \right\} dA, \tag{7.34}$$

where the second equation follows since the integrand in braces can be expanded to obtain

$$-y + \phi_{,x} + x\phi_{,xx} + x\phi_{,yy} = \phi_{,x} - y + x(\phi_{,xx} + \phi_{,yy}) = \phi_{,x} - y. \tag{7.35}$$

Since the integrand in Eq. 7.34 is in the form of the divergence of a vector field, we use the divergence theorem to obtain

$$\begin{aligned} F_x &= \mu\alpha \oint_C [n_1 x(-y + \phi_{,x}) + n_2 x(x + \phi_{,y})]\, ds \\ &= \mu\alpha \oint_C x[n_1(-y + \phi_{,x}) + n_2(x + \phi_{,y})]\, ds = 0, \end{aligned} \tag{7.36}$$

where the expression in brackets vanishes according to the boundary condition, Eq. 7.27. The two-dimensional form of the divergence theorem was used to obtain this result:

$$\int_A \nabla \cdot \mathbf{f}\, dA = \oint_C \mathbf{f} \cdot \mathbf{n}\, ds, \tag{7.37}$$

where $\mathbf{f}$ is a vector field defined over the two-dimensional area A bounded by the closed curve C. Similarly, $F_y = 0$, where F_y is the resultant force in the y direction.

The resultant moment on the top of the shaft is given by (Fig. 33)

$$M = \int_A (t_2 x - t_1 y)\, dA \tag{7.38}$$

$$= \mu\alpha \int_A [(\phi_{,y} + x)x - (\phi_{,x} - y)y]\, dA \tag{7.39}$$

$$= \mu\alpha \int_A [(x^2 + y^2) + \phi_{,y} x - \phi_{,x} y]\, dA, \tag{7.40}$$

where

$$I = \int_A (x^2 + y^2)\, dA \tag{7.41}$$

is the polar moment of inertia. Thus,

$$M = \mu\alpha \left[I + \int_A \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x} \right) dA \right] = \mu\alpha J, \tag{7.42}$$

where we define the *torsional constant* J as

$$J = I + \int_A \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x} \right) dA. \tag{7.43}$$

Note that, for a circular section, the warping function $\phi = 0$, $J = I$, and $M = \mu\alpha I$. J depends only on the geometry of the cross-section.

An alternative formula for J can also be derived. From the last equation,

$$J - I = \int_A (\phi_{,y} x - \phi_{,x} y)\, dA = \int_A \left[\frac{\partial}{\partial y}(\phi x) - \frac{\partial}{\partial x}(\phi y) \right] dA \tag{7.44}$$

$$= \oint_C (\phi x n_2 - \phi y n_1)\, ds = \oint_C (x n_2 - y n_1)\phi\, ds, \tag{7.45}$$

where the area integral is transformed into a contour integral using the two-dimensional form of the divergence theorem. Then, from Eq. 7.33b,

$$J - I = -\oint_C \frac{\partial \phi}{\partial n}\phi\, ds = -\oint_C \phi \nabla\phi \cdot \mathbf{n}\, ds = -\oint_C \phi\phi_{,i} n_i\, ds = -\int_A (\phi\phi_{,i})_{,i}\, dA, \tag{7.46}$$

where the last equation follows from the two-dimensional form of the divergence theorem. This integrand can be expanded to yield

$$J - I = -\int_A (\phi_{,i}\phi_{,i} + \phi\phi_{,ii})\, dA = -\int_A \nabla\phi \cdot \nabla\phi\, dA, \tag{7.47}$$

where $\phi_{,ii} = \nabla^2\phi = 0$. Thus, an alternative expression for the torsional constant J is

$$J = I - \int_A \nabla\phi \cdot \nabla\phi\, dA. \tag{7.48}$$

7.3 Uniqueness of Warping Function in Torsion Problem

The boundary value problem satisfied by the warping function ϕ in the torsion problem is

$$\begin{cases} \nabla^2\phi = 0 \ \text{ in } A, \\ \dfrac{\partial \phi}{\partial n} = n_1 y - n_2 x \ \text{ on } C. \end{cases} \tag{7.49}$$

As we saw in §6.1, uniqueness for this linear nonhomogeneous problem is equivalent to having uniqueness for the corresponding homogeneous problem

$$\begin{cases} \nabla^2\phi = 0 \ \text{ in } A, \\ \dfrac{\partial \phi}{\partial n} = 0 \ \text{ on } C. \end{cases} \tag{7.50}$$

That is, the difference of two solutions $\phi_1 - \phi_2$ of the nonhomogeneous problem satisfies the homogeneous problem.

Since ϕ satisfies Laplace's equation ($\phi_{,ii} = 0$), it follows that

$$0 = \int_A \phi\phi_{,ii}\, dA = \int_A [(\phi\phi_{,i})_{,i} - \phi_{,i}\phi_{,i}]\, dA = \oint_C \phi\phi_{,i} n_i\, ds - \int_A \nabla\phi \cdot \nabla\phi\, dA, \tag{7.51}$$

where $\phi_{,i} n_i = \nabla\phi \cdot \mathbf{n} = \partial\phi/\partial n = 0$ from the boundary condition. Thus,

$$\int_A |\nabla\phi|^2\, dA = 0, \tag{7.52}$$

which implies $\nabla\phi \equiv \mathbf{0}$ and hence $\phi =$ constant. That is, two solutions ϕ_1 and ϕ_2 of the nonhomogeneous problem can differ, at most, by a constant. Since the z displacement is proportional to the warping function, the nonuniqueness is equivalent to a translation in the z direction.

7.4 Existence of Warping Function in Torsion Problem

The boundary value problem satisfied by the warping function ϕ in the torsion problem is

$$\begin{cases} \nabla^2\phi = 0 \ \text{ in } A, \\ \dfrac{\partial\phi}{\partial n} = n_1 y - n_2 x \ \text{ on } C. \end{cases} \tag{7.53}$$

Since ϕ satisfies Laplace's equation, it follows that

$$0 = \int_A \nabla^2\phi\, dA = \int_A \nabla \cdot \nabla\phi\, dA = \oint_C \nabla\phi \cdot \mathbf{n}\, ds = \oint_C \frac{\partial\phi}{\partial n}\, ds. \tag{7.54}$$

That is, for a solution ϕ to exist, the integral of the boundary data must vanish. This is a *necessary* condition for existence. For the torsion problem, from Eq. 7.53b,

$$\oint_C \frac{\partial\phi}{\partial n}\, ds = \oint_C (n_1 y - n_2 x)\, ds = \int_A \left[\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x)\right] dA = 0, \tag{7.55}$$

as required, where the divergence theorem enabled us to transform the contour integral to an area integral. It is beginning to look as if a solution to the torsion problem exists.

Since the temperature field in a steady-state heat conduction problem also satisfies Laplace's equation, the boundary value problem satisfied by the warping function is similar to that satisfied by the temperature in a steady-state heat conduction problem where only heat flux boundary conditions are specified. Thus, the necessary condition

$$\oint_C \frac{\partial\phi}{\partial n}\, ds = 0 \tag{7.56}$$

for Eq. 7.53 to have a solution is analogous to requiring that the *net* heat flux across the boundary is zero in the heat transfer problem. That is, in the heat conduction problem where only heat flux boundary conditions are specified, a necessary condition for a steady-state solution to exist is that the net heat flux crossing the boundary is zero. A nonzero net heat flux would correspond physically to a net addition or subtraction of energy to the system, thus precluding steady-state temperatures.

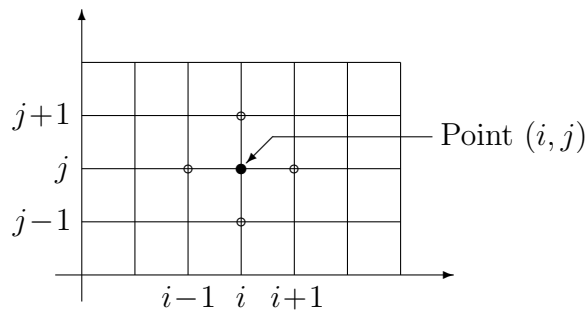


Figure 34: Finite Difference Grid on Rectangular Domain.

7.5 Some Properties of Harmonic Functions

Functions which satisfy Laplace's equation are said to be harmonic functions. Harmonic functions have two properties of interest:

1. A function which is harmonic inside a circular domain has a value at the center of the circle equal to the average of the values around the circle.

2. A harmonic function achieves it extreme values (maximum and minimum) on the boundary.

The first of these properties is clearly seen physically in steady-state heat conduction on a thin circular plate with a prescribed nonuniform boundary temperature, a problem for which the temperature satisfies Laplace's equation. Since all boundary points would have equal influence on the center point, the temperature at the center point must equal the average of the boundary temperatures.

There is also an interesting numerical analogue to the average value property of harmonic functions. If Laplace's equation is approximated numerically using central finite differences on the same uniform mesh in each direction (Fig. 34), the solution at a typical point (i, j) is the average of the four neighboring points:

$$\phi_{i,j} = (\phi_{i-1,j} + \phi_{i+1,j} + \phi_{i,j-1} + \phi_{i,j+1})/4. \tag{7.57}$$

This property can be used as the basis for a numerical solution using either direct or iterative approaches.

7.6 Prandtl Stress Function for Torsion

The torsion problem can also be formulated and solved in terms of a stress function, a potential function which, when differentiated, yields the stresses of interest. We recall from Eq. 7.24 that the stress tensor in the torsion problem is

$$\boldsymbol{\sigma} = \mu\alpha \begin{bmatrix} 0 & 0 & \phi_{,x} - y \\ 0 & 0 & \phi_{,y} + x \\ \phi_{,x} - y & \phi_{,y} + x & 0 \end{bmatrix}, \tag{7.58}$$

where ϕ is the warping function. The only stress components of interest are σ_{zx} and σ_{zy}, which must satisfy the equilibrium equation

$$\sigma_{zx,x} + \sigma_{zy,y} = 0. \tag{7.59}$$

We now define a *stress function* $\psi(x, y)$ such that

$$\sigma_{zx} = \psi_{,y}, \quad \sigma_{zy} = -\psi_{,x}. \tag{7.60}$$

With this definition of ψ, the equilibrium equation is automatically satisfied, since

$$\sigma_{zx,x} + \sigma_{zy,y} = \psi_{,yx} - \psi_{,xy} = 0. \tag{7.61}$$

Also, from Eq. 7.58, the stresses of interest are given by

$$\begin{cases} \sigma_{zx} = \mu\alpha(\phi_{,x} - y) = \psi_{,y}, \\ \sigma_{zy} = \mu\alpha(\phi_{,y} + x) = -\psi_{,x}. \end{cases} \tag{7.62}$$

We can eliminate ϕ from these equations by differentiating the first equation with respect to y, the second equation with respect to x, and subtracting the two results:

$$\psi_{,xx} + \psi_{,yy} = \mu\alpha(-\phi_{,yx} - 1 + \phi_{,xy} - 1) = -2\mu\alpha. \tag{7.63}$$

Thus, the stress function $\psi(x, y)$ satisfies the 2-D Poisson equation

$$\nabla^2\psi = -2\mu\alpha. \tag{7.64}$$

To determine the boundary condition on ψ, consider the curves $\psi(x, y) =$ constant. We can differentiate with respect to x to obtain

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0, \tag{7.65}$$

which implies that the tangent to the curve $\psi =$ constant has slope

$$\frac{dy}{dx} = \frac{-\frac{\partial\psi}{\partial x}}{\frac{\partial\psi}{\partial y}} = \frac{\sigma_{zy}}{\sigma_{zx}}. \tag{7.66}$$

Thus, we can define the tangent vector to the curve $\psi =$ constant as

$$\mathbf{T} = \sigma_{zx}\mathbf{e}_x + \sigma_{zy}\mathbf{e}_y. \tag{7.67}$$

On the other hand, the requirement for zero shear stress on the side of the shaft implies

$$0 = t_3 = \sigma_{zx}n_x + \sigma_{zy}n_y = \mathbf{T} \cdot \mathbf{n}. \tag{7.68}$$

That is, the curve $\psi =$ constant is perpendicular to the normal $\mathbf{n}$ on the boundary or, equivalently, the boundary is a curve $\psi =$ constant. Since ψ is a potential function (since the stresses are derivatives of ψ), a constant shift of ψ would not affect the stresses. Thus, for convenience, we choose $\psi = 0$ on the boundary.

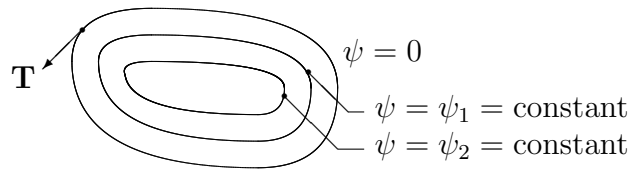


Figure 35: Stress Trajectories.

To summarize, the torsion problem requires solving Poisson's equation in the cross-section subject to a Dirichlet boundary condition:

$$\begin{cases} \nabla^2\psi = -2\mu\alpha \ \text{ in cross-section,} \\ \psi = 0 \ \text{ on boundary,} \end{cases} \tag{7.69}$$

where

$$\sigma_{zx} = \frac{\partial\psi}{\partial y}, \quad \sigma_{zy} = -\frac{\partial\psi}{\partial x}. \tag{7.70}$$

Note that

$$\nabla\psi = \frac{\partial\psi}{\partial x}\mathbf{e}_x + \frac{\partial\psi}{\partial y}\mathbf{e}_y = -\sigma_{zy}\mathbf{e}_x + \sigma_{zx}\mathbf{e}_y, \tag{7.71}$$

from which it follows that

$$\nabla\psi \cdot \mathbf{T} = -\sigma_{zy}\sigma_{zx} + \sigma_{zx}\sigma_{zy} = 0. \tag{7.72}$$

That is, the stress function ψ changes fastest in the direction perpendicular to the curves $\psi =$ constant. The $\psi =$ constant curves (for different values of the constant) are called *stress trajectories* (Fig. 35).

To compute the torque (end moment) M at the top of the shaft, we recall Eq. 7.38 and Fig. 33:

$$M = \int_A (t_2 x - t_1 y)\, dA = \int_A (\sigma_{zy} x - \sigma_{zx} y)\, dA = -\int_A (\psi_{,x} x + \psi_{,y} y)\, dA. \tag{7.73}$$

The first term of this last integral can be written in the form

$$-\int_A \psi_{,x} x\, dA = -\int \left[\int x \frac{\partial\psi}{\partial x}\, dx\right] dy, \tag{7.74}$$

the inner integral of which can be integrated by parts. Since

$$d(uv) = (du)v + u(dv), \tag{7.75}$$

integration by parts states that

$$\int_a^b u\, dv = (uv)\Big|_a^b - \int_a^b v\, du, \tag{7.76}$$

where the first term of the right-hand side is the boundary term. Thus, in Eq. 7.74, we make the associations

$$u = x, \quad dv = \frac{\partial\psi}{\partial x}\, dx, \tag{7.77}$$

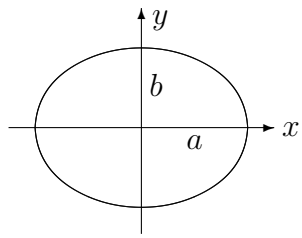


Figure 36: Elliptical Cylinder.

to obtain

$$-\int_A \psi_{,x} x \, dA = -\int \left[\psi x \Big|_{x=x_1}^{x=x_2} - \int \psi \, dx \right] dy = \int_A \psi \, dA, \tag{7.78}$$

where the boundary term vanishes since $\psi = 0$ on the boundary. Similarly, we can show that the second term in Eq. 7.73 is also given by

$$\int_A \psi \, dA.$$

Thus, the end moment (torque) is given by

$$M = 2 \int_A \psi \, dA. \tag{7.79}$$

Since, from Eq. 7.42,

$$M = \mu \alpha J, \tag{7.80}$$

where J is the torsional constant, J can be expressed in terms of the stress function as

$$J = \frac{2}{\mu \alpha} \int_A \psi \, dA. \tag{7.81}$$

7.7 Torsion of Elliptical Cylinder

Consider the ellipse (Fig. 36) with semi-axes a and b in the x and y directions, respectively:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{7.82}$$

For torsion of a few cases where the boundary is a simple geometrical shape, closed form analytical solutions of the torsion equation can be obtained directly from the equation of the boundary. We attempt to find a stress function in the form

$$\psi = c \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right), \tag{7.83}$$

where c is an unknown constant, and $\psi = 0$ on the boundary, as required. Since ψ must satisfy Poisson's equation,

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = c \left(\frac{2}{a^2} + \frac{2}{b^2} \right) = -2\mu\alpha. \tag{7.84}$$

Thus,

$$c = -\mu\alpha / \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \tag{7.85}$$

and

$$\psi = \frac{\mu\alpha a^2 b^2}{a^2 + b^2} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right). \tag{7.86}$$

The torsional constant J is given by Eq. 7.81 as

$$J = \frac{2a^2b^2}{a^2+b^2} \int_A \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) dA = \frac{2a^2b^2}{a^2+b^2} \left(A - \frac{I_y}{a^2} - \frac{I_x}{b^2} \right), \tag{7.87}$$

where A is the area of the elliptical cross-section, and I_y and I_x are the area moments of inertia of the cross-section with respect to the y and x axes, respectively, and

$$A = \pi ab, \quad I_y = \int_A x^2 \, dA = \frac{\pi}{4} ba^3, \quad I_x = \int_A y^2 \, dA = \frac{\pi}{4} ab^3. \tag{7.88}$$

Hence,

$$J = \frac{2a^2b^2}{a^2+b^2} \left(\pi ab - \frac{\pi}{4} ba - \frac{\pi}{4} ab \right) = \frac{\pi a^3 b^3}{a^2+b^2}. \tag{7.89}$$

We compute the shear stresses from Eq. 7.70 as

$$\sigma_{zx} = \frac{\partial \psi}{\partial y} = \frac{\mu\alpha a^2 b^2}{a^2+b^2} \left(\frac{-2y}{b^2} \right) = -\frac{2\mu\alpha a^2 y}{a^2+b^2}, \tag{7.90}$$

$$\sigma_{zy} = -\frac{\partial \psi}{\partial x} = \frac{-\mu\alpha a^2 b^2}{a^2+b^2} \left(\frac{-2x}{a^2} \right) = \frac{2\mu\alpha b^2 x}{a^2+b^2}. \tag{7.91}$$

These expressions imply that the maximum shear stresses occur on the boundary:

$$\sigma_{zx}\Big|_{\max} = \sigma_{zx}\Big|_{y=b} = -\frac{2\mu\alpha ab}{a^2+b^2} a, \tag{7.92}$$

$$\sigma_{zy}\Big|_{\max} = \sigma_{zy}\Big|_{x=a} = \frac{2\mu\alpha ab}{a^2+b^2} b. \tag{7.93}$$

If $a > b$, the maximum shear stress occurs at the point $(0, b)$. That is, for this case, the maximum shear stress occurs at the closest boundary point. In general, for other shapes, it can be shown that, if the boundary of the shaft is convex, the maximum shear occurs at the boundary point which is closest to the centroid of the cross-section.

7.8 Torsion of Rectangular Bars: Warping Function

The torsion problem for a rectangular bar can be solved in terms of either the warping function or the stress function. We first consider a warping function approach for solving the torsion problem for rectangular bars. Consider the rectangle of sides $2a$ and $2b$ in the x and y directions, respectively (Fig. 37). From Eq. 7.33, the warping function for this problem satisfies the boundary value problem

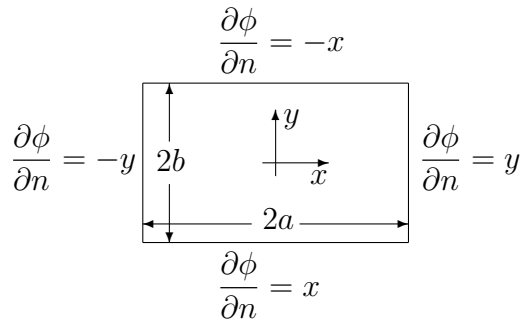


Figure 37: Torsion of Rectangular Bar.

$$\begin{cases} \nabla^2\phi = 0 \quad (-a < x < a,\ -b < y < b), \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\phi}{\partial x} = y \quad \text{on } x = a, \\ \dfrac{\partial\phi}{\partial n} = -\dfrac{\partial\phi}{\partial x} = -y \quad \text{on } x = -a, \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\phi}{\partial y} = -x \quad \text{on } y = b, \\ \dfrac{\partial\phi}{\partial n} = -\dfrac{\partial\phi}{\partial y} = x \quad \text{on } y = -b. \end{cases} \tag{7.94}$$

We observe from Fig. 37 that the boundary conditions on $x = \pm a$ are odd functions of y. Also, the boundary conditions on $y = \pm b$ are odd functions of x. Moreover, a mirror image reflection of the problem through the y-axis yields a new problem whose solution is the negative of the original problem. Similarly, a mirror image reflection of the problem through the x-axis yields a new problem whose solution is the negative of the original problem. Thus, the solution of Eq. 7.94 must be antisymmetric in both x and y, in which case an equivalent boundary value problem can be defined on the domain $0 < x < a$, $0 < y < b$ (Fig. 38):

$$\begin{cases} \nabla^2\phi = 0 \quad (0 < x < a,\ 0 < y < b), \\ \phi = 0 \quad \text{on } x = 0 \ \text{ and } \ y = 0, \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\phi}{\partial x} = y \quad \text{on } x = a, \\ \dfrac{\partial\phi}{\partial n} = \dfrac{\partial\phi}{\partial y} = -x \quad \text{on } y = b. \end{cases} \tag{7.95}$$

This problem, which has nonhomogeneous boundary conditions on adjacent boundaries, can be simplified somewhat by transforming to a new variable w such that[22]

$$\phi(x, y) = xy - w(x, y), \tag{7.96}$$

where

$$\frac{\partial\phi}{\partial x} = y - \frac{\partial w}{\partial x}, \quad \frac{\partial\phi}{\partial y} = x - \frac{\partial w}{\partial y}. \tag{7.97}$$

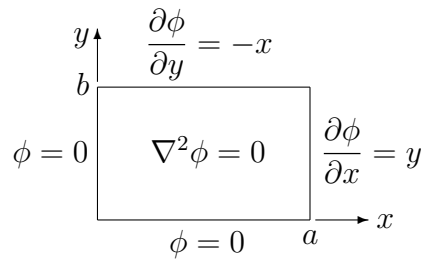


Figure 38: Torsion of Rectangular Bar Using Symmetry.

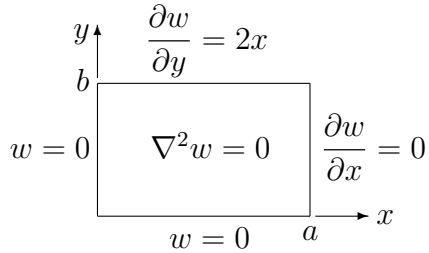


Figure 39: Torsion of Rectangular Bar Using Transformed Variable.

Thus, the boundary value problem satisfied by w is

$$\begin{cases} \nabla^2 w = 0 \quad (0 < x < a,\ 0 < y < b), \\ w = 0 \quad \text{on } x = 0 \ \text{ and } \ y = 0, \\ \dfrac{\partial w}{\partial x} = 0 \quad \text{on } x = a, \\ \dfrac{\partial w}{\partial y} = 2x \quad \text{on } y = b, \end{cases} \tag{7.98}$$

as illustrated in Fig. 39.

We attempt a solution of Eq. 7.98 using the technique of separation of variables, and look for solutions of the form

$$w(x, y) = X(x)Y(y). \tag{7.99}$$

The substitution of this equation into Eq. 7.98a leads to

$$X''Y + XY'' = 0 \tag{7.100}$$

or

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}, \tag{7.101}$$

implying that, for all x and y, a function of x is equal to a function of y. Thus, each function must be a constant:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2, \tag{7.102}$$

where the constant λ^2 is referred to as the *separation constant*. The constant is taken as negative, since otherwise we would be unable to satisfy the boundary conditions.

Eq. 7.102 thus separates into two ordinary differential equations

$$\begin{cases} X'' + \lambda^2 X = 0, \\ Y'' - \lambda^2 Y = 0, \end{cases} \tag{7.103}$$

the general solutions of which are

$$\begin{cases} X(x) = A \sin \lambda x + B \cos \lambda x, \\ Y(y) = C \sinh \lambda y + D \cosh \lambda y, \end{cases} \tag{7.104}$$

where A, B, C, and D are constants. However, since $X(x)$ and $Y(y)$ must be odd functions (as implied also by the zero boundary conditions $X(0) = 0$ and $Y(0) = 0$), if follows that $B = D = 0$.

The boundary condition, Eq. 7.98c, implies

$$0 = X'(a) = A\lambda \cos \lambda a. \tag{7.105}$$

Thus, either $\lambda = 0$ or $\lambda = \lambda_n$, where

$$\lambda_n a = (2n-1)\frac{\pi}{2}, \quad n = 1, 2, 3, \ldots . \tag{7.106}$$

We discard the case $\lambda = 0$, since Eq. 7.103a (with $\lambda = 0$) and the boundary conditions $X(0) = 0$ and $X'(a) = 0$ imply $X(x) \equiv 0$ for this case. Thus, from Eq. 7.99, the solution must be of the form $\sin \lambda_n x \, \sinh \lambda_n y$. Since functions of this type satisfy the partial differential equation and the three homogeneous boundary conditions for all n, the most general form of the solution is obtained by taking a linear combination of these functions:

$$w(x, y) = \sum_{n=1}^{\infty} A_n \sin \lambda_n x \, \sinh \lambda_n y. \tag{7.107}$$

This series satisfies the partial differential equation ($\nabla^2 w = 0$) and the three homogeneous boundary conditions. If we can force the series to satisfy the nonhomogeneous boundary condition on $y = b$, then we have solved the problem, Eq. 7.98, since it is well-posed. *Well-posed* means that the solution exists, is unique, and depends continuously on the data (the various parameters of the problem, including geometry, material properties, boundary conditions, and initial conditions, if any).

The boundary condition at $y = b$ implies

$$2x = \sum_{n=1}^{\infty} A_n \lambda_n \sin \lambda_n x \, \cosh \lambda_n b. \tag{7.108}$$

If we now multiply both sides of this equation by $\sin \lambda_m x$ (for fixed m an integer), and integrate from 0 to a, we obtain

$$\int_0^a 2x \sin \lambda_m x \, dx = \sum_{n=1}^{\infty} A_n \lambda_n \cosh \lambda_n b \int_0^a \sin \lambda_n x \, \sin \lambda_m x \, dx, \tag{7.109}$$

where

$$\int_0^a \sin\lambda_n x \, \sin\lambda_m x \, dx = \begin{cases} 0, & n \neq m, \\ a/2, & n = m, \end{cases} \tag{7.110}$$

and

$$\int_0^a x \sin\lambda_m x \, dx = \frac{1}{\lambda_m^2}(-1)^{m+1}. \tag{7.111}$$

Eq. 7.110 implies that only one term (the term $n = m$) remains in the series in Eq. 7.109. Thus,

$$A_n = \frac{4a^2(-1)^{n+1}}{(\lambda_n a)^3 \cosh\lambda_n b} = \frac{32a^2(-1)^{n+1}}{\pi^3(2n-1)^3\cosh\lambda_n b}, \tag{7.112}$$

where $\lambda_n a$ is given by Eq. 7.106. The warping function for this problem is therefore, from Eqs. 7.96 and Eq. 7.107,

$$\phi(x,y) = xy - \frac{32a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\cdot\frac{1}{\cosh\lambda_n b}\sin\lambda_n x \, \sinh\lambda_n y. \tag{7.113}$$

The torsional constant is given by Eq. 7.43 as

$$J = I + \int_{-b}^{b}\int_{-a}^{a}\left(x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x}\right)dx\,dy. \tag{7.114}$$

We make the following observations about this integral:

1. ϕ is odd in x and odd in y.
2. $\partial\phi/\partial y$ is odd in x and even in y.
3. $\partial\phi/\partial x$ is even in x and odd in y.
4. $x\,\partial\phi/\partial y$ is even in x and even in y.
5. $y\,\partial\phi/\partial x$ is even in x and even in y.

Thus, the integrand in Eq. 7.114 is an even function of both x and y. Hence,

$$J = I + 4\int_0^b\int_0^a\left(x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x}\right)dx\,dy, \tag{7.115}$$

where $I = I_x + I_y$ and

$$I_x = \int_A y^2\,dA = \frac{4}{3}ab^3, \tag{7.116}$$

$$I_y = \int_A x^2\,dA = \frac{4}{3}a^3b. \tag{7.117}$$

Thus,

$$J = I_x + I_y + 4\int_0^b \int_0^a \left\{ x\left[x - \frac{32a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\cdot\frac{\lambda_n}{\cosh\lambda_n b}\sin\lambda_n x\,\cosh\lambda_n y\right]\right.$$

$$\left. -y\left[y - \frac{32a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\cdot\frac{\lambda_n}{\cosh\lambda_n b}\cos\lambda_n x\,\sinh\lambda_n y\right]\right\} dx\,dy \tag{7.118}$$

$$= I_x + I_y + I_y - I_x - \frac{128a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\cdot\frac{1}{\cosh\lambda_n b}\left(\int_0^a x\sin\lambda_n x\,dx\right)\sinh\lambda_n b$$

$$+\frac{128a^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\cdot\frac{1}{\cosh\lambda_n b}\sin\lambda_n a\int_0^b y\sinh\lambda_n y\,dy, \tag{7.119}$$

where the first integral is given by Eq. 7.111 and

$$\int_0^b y\sinh\lambda_n y\,dy = -\frac{1}{\lambda_n^2}\sinh\lambda_n b + \frac{b}{\lambda_n}\cosh\lambda_n b. \tag{7.120}$$

Thus, after some rearranging,

$$J = 2I_y + \frac{128a^3b}{\pi^3}\left[\sum_{n=1}^{\infty}\frac{2}{(2n-1)^4\pi} - \sum_{n=1}^{\infty}\frac{2a\tanh\lambda_n b}{b(2n-1)^5}\left(\frac{2}{\pi}\right)^2\right] \tag{7.121}$$

$$= \frac{8}{3}a^3b + \frac{256a^3b}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} - \frac{1024a^4}{\pi^5}\sum_{n=1}^{\infty}\frac{\tanh\lambda_n b}{(2n-1)^5} \tag{7.122}$$

$$= \frac{8}{3}a^3b\left[1 + \frac{96}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} - \frac{384a}{\pi^5 b}\sum_{n=1}^{\infty}\frac{\tanh\lambda_n b}{(2n-1)^5}\right], \tag{7.123}$$

where it can be shown[22] that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}. \tag{7.124}$$

We thus obtain the final result

$$J = \frac{16}{3}a^3b\left[1 - \frac{192a}{\pi^5 b}\sum_{n=1}^{\infty}\frac{\tanh\lambda_n b}{(2n-1)^5}\right] = \kappa a^3 b. \tag{7.125}$$

The constant κ is tabulated below for several values of b/a:

b/a	κ
1.0	2.249
1.2	2.658
1.5	3.132
2.0	3.659
2.5	3.990
3.0	4.213
4.0	4.493
5.0	4.661
10.0	4.997
∞	5.333

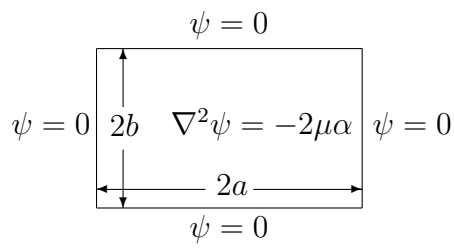


Figure 40: Torsion of Rectangular Bar Using Stress Function.

Note that the series in the last equation can be written in the form

$$\sum_{n=1}^{\infty} \frac{\tanh \lambda_n b}{(2n-1)^5} = \tanh \frac{\pi b}{2a} + \sum_{n=2}^{\infty} \frac{\tanh \lambda_n b}{(2n-1)^5}, \tag{7.126}$$

where

$$\sum_{n=2}^{\infty} \frac{\tanh \lambda_n b}{(2n-1)^5} < \sum_{n=2}^{\infty} \frac{1}{(2n-1)^5} \approx 0.00452 \tag{7.127}$$

and, for $b \geq a$,

$$\tanh \frac{\pi b}{2a} \geq 0.917 \tag{7.128}$$

imply that, with only the first term of the series, we obtain the sum to within about 0.5%. Thus, a practical approximation to the torsional constant is

$$J \approx \frac{16}{3} a^3 b \left(1 - \frac{192a}{\pi^5 b} \tanh \frac{\pi b}{2a} \right). \tag{7.129}$$

7.9 Torsion of Rectangular Bars: Stress Function

Here we solve the torsion problem for a rectangular cross-section using the stress function. Consider the rectangle of sides $2a$ and $2b$ in the x and y directions, respectively, where we locate the coordinate origin at the center of the rectangle (Fig. 40). According to Eq. 7.69, the stress function satisfies the boundary value problem

$$\begin{cases} \nabla^2\psi = -2\mu\alpha \quad (-a < x < a, \ -b < y < b), \\ \psi = 0 \quad \text{on boundary}, \end{cases} \tag{7.130}$$

Since the right-hand side of the partial differential equation satisfied by the stress function is a constant, we can transform from the Poisson equation to the Laplace equation by expressing the stress function in the form

$$\psi(x, y) = \mu\alpha(a^2 - x^2) + \mu\alpha v(x, y), \tag{7.131}$$

in which case the boundary value problem satisfied by v is

$$\begin{cases} \nabla^2 v = 0 \quad (-a < x < a, \ -b < y < b), \\ v(\pm a, y) = 0, \\ v(x, \pm b) = x^2 - a^2, \end{cases} \tag{7.132}$$

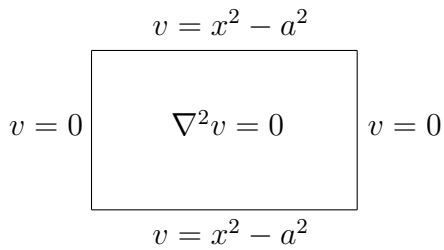


Figure 41: Torsion of Rectangular Bar Using Transformed Stress Function.

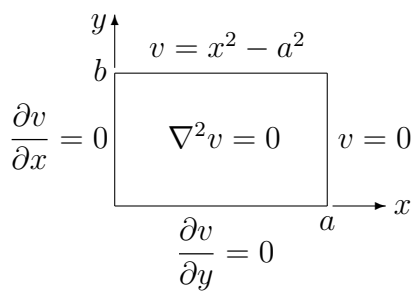


Figure 42: Torsion of Rectangular Bar Using Stress Function and Symmetry.

as illustrated in Fig. 41.

We observe that the domain is symmetric, and the boundary conditions are even functions of x and y and symmetric with respect to the two coordinate axes. Hence, the solution of Eq. 7.132 must exhibit the same symmetry and consist of even functions of x and y. Thus, an equivalent boundary value problem can be defined on the domain $0 < x < a$, $0 < y < b$ (Fig. 42):

$$\begin{cases} \nabla^2 v = 0 \quad (0 < x < a,\ 0 < y < b), \\ \dfrac{\partial v}{\partial x} = 0 \quad \text{on } x = 0, \\ \dfrac{\partial v}{\partial y} = 0 \quad \text{on } y = 0, \\ v(a, y) = 0, \\ v(x, b) = f(x) = x^2 - a^2. \end{cases} \tag{7.133}$$

We attempt a solution of Eq. 7.133 using the technique of separation of variables, and look for solutions of the form

$$v(x, y) = X(x)Y(y). \tag{7.134}$$

The substitution of this equation into Eq. 7.133a leads to

$$X''Y + XY'' = 0 \tag{7.135}$$

or

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}, \tag{7.136}$$

implying that, for all x and y, a function of x is equal to a function of y. Thus, each function must be a constant:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda^2, \tag{7.137}$$

where the constant λ^2 is the separation constant. The sign of the constant must be negative, since otherwise we would be unable to satisfy the boundary conditions.

Eq. 7.137 thus separates into two ordinary differential equations

$$\begin{cases} X'' + \lambda^2 X = 0, \\ Y'' - \lambda^2 Y = 0, \end{cases} \tag{7.138}$$

the general solutions of which are

$$\begin{cases} X(x) = A \sin \lambda x + B \cos \lambda x, \\ Y(y) = C \sinh \lambda y + D \cosh \lambda y, \end{cases} \tag{7.139}$$

where A, B, C, and D are constants, and

$$\begin{cases} X'(x) = A\lambda \cos \lambda x - B\lambda \sin \lambda x, \\ Y'(y) = C\lambda \cosh \lambda y + D\lambda \sinh \lambda y. \end{cases} \tag{7.140}$$

The zero slope condition at $x = 0$ implies

$$0 = X'(0) = A\lambda, \tag{7.141}$$

which implies either $A = 0$ or $\lambda = 0$. Similarly, the zero slope condition at $y = 0$ implies

$$0 = Y'(0) = C\lambda, \tag{7.142}$$

which implies either $C = 0$ or $\lambda = 0$. We discard the case $\lambda = 0$, since Eq. 7.138a (with $\lambda = 0$) and the boundary conditions $X'(0) = 0$ and $X(a) = 0$ imply $X(x) \equiv 0$ for this case. Thus, $A = C = 0$.

The homogeneous boundary condition, Eq. 7.133d, implies

$$0 = X(a) = B \cos \lambda a, \tag{7.143}$$

which implies that $\lambda = \lambda_n$, where

$$\lambda_n a = (2n - 1)\frac{\pi}{2} \quad (n = 1, 2, 3, \ldots). \tag{7.144}$$

Thus, from Eq. 7.134, the solution must be of the form $\cos \lambda_n x \cosh \lambda_n y$. Since functions of this type satisfy the partial differential equation and the three homogeneous boundary conditions for all n, the most general form of the solution is obtained by taking a linear combination of these functions:

$$v(x, y) = \sum_{n=1}^{\infty} A_n \cos \lambda_n x \cosh \lambda_n y. \tag{7.145}$$

This series satisfies the partial differential equation ($\nabla^2 v = 0$) and the three homogeneous boundary conditions. If we can force the series to satisfy the nonhomogeneous boundary condition on $y = b$, then we have solved the problem, Eq. 7.133, since it is well-posed.

The boundary condition at $y = b$ implies

$$f(x) = x^2 - a^2 = \sum_{n=1}^{\infty} A_n \cos \lambda_n x \, \cosh \lambda_n b. \tag{7.146}$$

If we now multiply both sides of this equation by $\cos \lambda_m x$ and integrate from 0 to a, we obtain

$$\int_0^a f(x) \cos \lambda_m x \, dx = \sum_{n=1}^{\infty} A_n \cosh \lambda_n b \int_0^a \cos \lambda_n x \, \cos \lambda_m x \, dx, \tag{7.147}$$

where

$$\int_0^a f(x) \cos \lambda_m x \, dx = \int_0^a (x^2 - a^2) \cos \lambda_m x \, dx = \frac{2(-1)^m}{\lambda_m^3}, \tag{7.148}$$

$$\int_0^a \cos \lambda_n x \, \cos \lambda_m x \, dx = \begin{cases} 0, & n \neq m, \\ a/2, & n = m, \end{cases} \tag{7.149}$$

and $\sin \lambda_n a = (-1)^{n+1}$. Eq. 7.149 implies that only one term (the term $n = m$) remains in the series in Eq. 7.147, so that

$$\frac{2(-1)^n}{\lambda_n^3} = A_n \cosh \lambda_n b \left(\frac{a}{2}\right) \tag{7.150}$$

or

$$A_n = \frac{4(-1)^n}{a\lambda_n^3 \cosh \lambda_n b} = \frac{32a^2(-1)^n}{(2n-1)^3 \pi^3 \cosh \lambda_n b}, \tag{7.151}$$

where $\lambda_n a$ is given by Eq. 7.144. The stress function for this problem is therefore, from Eqs. 7.131 and Eq. 7.145,

$$\psi(x, y) = \mu\alpha(a^2 - x^2) + \mu\alpha \sum_{n=1}^{\infty} A_n \cos \lambda_n x \, \cosh \lambda_n y. \tag{7.152}$$

The torsional constant is given by Eq. 7.81 as

$$J = \frac{2}{\mu\alpha} \int_A \psi \, dA. \tag{7.153}$$

Since the integrand ψ is an even function of x and y,

$$J = \frac{8}{\mu\alpha}\int_0^b\int_0^a \psi\,dx\,dy \tag{7.154}$$

$$= 8\int_0^b\int_0^a \left[a^2 - x^2 + \sum_{n=1}^{\infty} A_n \cos\lambda_n x\,\cosh\lambda_n y\right] dx\,dy \tag{7.155}$$

$$= 8a^3b - \frac{8}{3}a^3b + 8\sum_{n=1}^{\infty} A_n \frac{\sin\lambda_n a}{\lambda_n}\cdot\frac{\sinh\lambda_n b}{\lambda_n} \tag{7.156}$$

$$= \frac{16}{3}a^3b + \sum_{n=1}^{\infty}\frac{256a^2(-1)^n}{(2n-1)^3\pi^3\cosh\lambda_n b}\cdot\frac{(-1)^{n+1}4a^2}{(2n-1)^2\pi^2}\cdot\sinh\lambda_n b \tag{7.157}$$

$$= \frac{16}{3}a^3b - a^4\left(\frac{4}{\pi}\right)^5\sum_{n=1}^{\infty}\frac{\tanh\lambda_n b}{(2n-1)^5}, \tag{7.158}$$

which agrees with the result, Eq. 7.125, derived using the warping function.

8 Plane Deformation

8.1 Plane Strain

A body is said to be in the state of *plane strain*, parallel to the xy-plane, if

$$\begin{cases} u_1 = u_1(x_1, x_2), \\ u_2 = u_2(x_1, x_2), \\ u_3 = 0. \end{cases} \tag{8.1}$$

These are the basic assumptions of plane strain. Such a deformation can occur only for cylindrical bodies (where the cross-section at each z is independent of z) of

1. infinite length, or
2. finite length with ends fixed in the z direction.

Under plane strain conditions, the strain-displacement equations,

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{8.2}$$

imply

$$\varepsilon_{13} = \varepsilon_{31} = \varepsilon_{23} = \varepsilon_{32} = \varepsilon_{33} = 0 \tag{8.3}$$

or

$$\varepsilon_{3i} = 0, \quad i = 1, 2, 3. \tag{8.4}$$

The constraint in the z direction, $u_3 = 0$, requires a force of constraint (reaction), which can, for an isotropic material, be determined from inverse Hooke's law, Eq. 5.25:

$$\varepsilon_{33} = -\nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}. \tag{8.5}$$

Thus, with $\varepsilon_{33} = 0$, we find that, for plane strain,

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}). \tag{8.6}$$

That is, for plane strain, σ_{33} is not independent, but determined from σ_{11} and σ_{22}.

We now consider the consequences of the equilibrium equations for plane strain. Since the displacements u_1 and u_2 depend only on x_1 and x_2, if follows that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ also depend only on x_1 and x_2:

$$\begin{cases} \sigma_{11} = \sigma_{11}(x_1, x_2), \\ \sigma_{22} = \sigma_{22}(x_1, x_2), \\ \sigma_{12} = \sigma_{12}(x_1, x_2), \\ \sigma_{33} = \nu(\sigma_{11} + \sigma_{22}), \\ \sigma_{13} = 0, \\ \sigma_{23} = 0. \end{cases} \tag{8.7}$$

The equations of static equilibrium are

$$\sigma_{ij,j} + \rho f_i = 0. \tag{8.8}$$

Thus,

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} + \rho f_1 = 0 \quad \Rightarrow \quad f_1 = f_1(x_1, x_2), \\ \sigma_{21,1} + \sigma_{22,2} + \rho f_2 = 0 \quad \Rightarrow \quad f_2 = f_2(x_1, x_2), \\ \rho f_3 = 0 \quad \Rightarrow \quad f_3 = 0. \end{cases} \tag{8.9}$$

That is, in plane strain, there can be no z-component of body force, and the two in-plane components depend only on x_1 and x_2.

Of the six strain compatibility equations, Eqs. 2.80–2.85, there is only one nontrivial equation,

$$2\varepsilon_{12,12} = \varepsilon_{11,22} + \varepsilon_{22,11}, \tag{8.10}$$

since all terms in the other five equations involve either a zero component of strain (ε_{3i}) or a derivative with respect to x_3.

8.2 Plane Stress

A body is said to be in a state of *plane stress* parallel to the xy-plane when

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0, \tag{8.11}$$

and the other components of stress are independent of z. These are the basic assumptions of plane stress.

From Hooke's law for an isotropic material,

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}. \tag{8.12}$$

Thus,

$$\begin{cases} \sigma_{13} = 0 \quad \Rightarrow \quad \varepsilon_{13} = 0, \\ \sigma_{23} = 0 \quad \Rightarrow \quad \varepsilon_{23} = 0, \\ \sigma_{33} = 0 \quad \Rightarrow \quad 0 = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22}) + (\lambda + 2\mu)\varepsilon_{33}, \end{cases} \tag{8.13}$$

or

$$\varepsilon_{33} = -\frac{\lambda}{\lambda + 2\mu}(\varepsilon_{11} + \varepsilon_{22}) = -\frac{\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22}), \tag{8.14}$$

where Eq. 5.15 was used. That is, even though the stress $\sigma_{33} = 0$, the strain $\varepsilon_{33} \neq 0$. The physical situation corresponding to plane stress would be a thin plate with no stress on the faces. A plate is considered *thin* if the thickness is small compared to the lateral dimensions in the xy-plane.

The strain-stress equation for an isotropic material,

$$\varepsilon_{33} = -\nu\frac{\sigma_{11}}{E} - \nu\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}, \tag{8.15}$$

also implies

$$\varepsilon_{33} = -\frac{\nu}{E}(\sigma_{11} + \sigma_{22}) \tag{8.16}$$

since $\sigma_{33} = 0$.

The equations of static equilibrium are

$$\sigma_{ij,j} + \rho f_i = 0. \tag{8.17}$$

Thus,

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} + \rho f_1 = 0, \\ \sigma_{21,1} + \sigma_{22,2} + \rho f_2 = 0, \\ \qquad\qquad\qquad \rho f_3 = 0, \end{cases} \tag{8.18}$$

where the last of these equations implies $f_3 = 0$. We note that the equilibrium equations for plane strain (Eq. 8.9) and plane stress (Eq. 8.18) look the same. However, in plane strain, the solution variables ($\mathbf{u}$, $\boldsymbol{\sigma}$, and $\boldsymbol{\varepsilon}$) are independent of z; in plane stress, the solution may depend on z, since $\varepsilon_{33} \neq 0$ implies $u_3 \neq 0$.

In plane stress, there is also only one nontrivial compatibility equation,

$$2\varepsilon_{12,12} = \varepsilon_{11,22} + \varepsilon_{22,11}. \tag{8.19}$$

Even though ε_{33} is nonzero, it does not influence the solution and can be recovered after a problem is solved in terms of the in-plane stress or strain components (e.g., Eq. 8.14 or Eq. 8.16).

8.3 Formal Equivalence Between Plane Stress and Plane Strain

In general three-dimensional elasticity, inverse Hooke's law for an isotropic material is, from Eqs. 5.23–5.25,

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}(\sigma_{yy} + \sigma_{zz}), \tag{8.20}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}), \tag{8.21}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}). \tag{8.22}$$

In plane stress, where $\sigma_{zz} = 0$,

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy}, \tag{8.23}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\sigma_{xx}, \tag{8.24}$$

or, inversely,

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}), \tag{8.25}$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}(\nu\varepsilon_{xx} + \varepsilon_{yy}). \tag{8.26}$$

The third normal component of strain can be determined once the two-dimensional problem is solved:

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) = -\frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}). \tag{8.27}$$

Thus, in plane stress, ε_{zz} is a Poisson's ratio effect that can be determined after the in-plane solution has been obtained. The in-plane solution is not influenced by ε_{zz}.

On the other hand, in plane strain ($\varepsilon_{zz} = 0$),

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \tag{8.28}$$

implies

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy} - \frac{\nu^2}{E}(\sigma_{xx} + \sigma_{yy}) = \frac{1-\nu^2}{E}\sigma_{xx} - \frac{\nu(1+\nu)}{E}\sigma_{yy}, \tag{8.29}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\sigma_{xx} - \frac{\nu^2}{E}(\sigma_{xx} + \sigma_{yy}) = \frac{1-\nu^2}{E}\sigma_{yy} - \frac{\nu(1+\nu)}{E}\sigma_{xx}. \tag{8.30}$$

To obtain the "equivalence" between plane strain and plane stress, we put the plane strain relations in the same form as those for plane stress:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{\bar{E}} - \frac{\bar{\nu}}{\bar{E}}\sigma_{yy}, \tag{8.31}$$

where $\bar{E}$ and $\bar{\nu}$ are the constants to be used in a plane stress solution to simulate plane strain. A comparison of this equation with Eq. 8.29 yields

$$\bar{E} = \frac{E}{1-\nu^2} \tag{8.32}$$

and

$$\frac{\bar{\nu}}{\bar{E}} = \frac{\nu(1+\nu)}{E}. \tag{8.33}$$

If we now substitute Eq. 8.32 into Eq. 8.33, we obtain

$$\bar{\nu} = \frac{\nu(1+\nu)}{E} \cdot \frac{E}{1-\nu^2} = \frac{\nu}{1-\nu}. \tag{8.34}$$

Thus, plane stress results in the plane can be converted to plane strain if E and ν are replaced by

$$\bar{E} = \frac{E}{1-\nu^2}, \quad \bar{\nu} = \frac{\nu}{1-\nu}, \tag{8.35}$$

where E and ν are the actual material properties.

8.4 Compatibility Equation in Terms of Stress

We recall that, for two-dimensional problems, we have the single nontrivial compatibility equation

$$2\varepsilon_{12,12} = \varepsilon_{11,22} + \varepsilon_{22,11}, \tag{8.36}$$

where, from Hooke's law for an isotropic material,

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}\sigma_{22}, \tag{8.37}$$

$$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu}{E}\sigma_{11}, \tag{8.38}$$

$$2\varepsilon_{12} = \frac{\sigma_{12}}{\mu} = \frac{2(1+\nu)}{E}\sigma_{12}. \tag{8.39}$$

These equations also apply to plane strain with suitable interpretation of the elastic constants (Eq. 8.35). The substitution of Hooke's law into Eq. 8.36 yields

$$\frac{2(1+\nu)\sigma_{12,12}}{E} = \frac{\sigma_{11,22} - \nu\sigma_{22,22}}{E} + \frac{\sigma_{22,11} - \nu\sigma_{11,11}}{E} \tag{8.40}$$

or

$$2(1+\nu)\sigma_{12,12} = \sigma_{11,22} - \nu\sigma_{22,22} + \sigma_{22,11} - \nu\sigma_{11,11}. \tag{8.41}$$

On the other hand, the equations of static equilibrium with zero body force,

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} = 0, \\ \sigma_{21,1} + \sigma_{22,2} = 0, \end{cases} \tag{8.42}$$

imply

$$\sigma_{12,12} = -\sigma_{11,11} = -\sigma_{22,22} \tag{8.43}$$

or

$$2\sigma_{12,12} = -\sigma_{11,11} - \sigma_{22,22}. \tag{8.44}$$

We now substitute this result into Eq. 8.41 to obtain

$$-(1+\nu)(\sigma_{11,11} + \sigma_{22,22}) = \sigma_{11,22} + \sigma_{22,11} - \nu(\sigma_{11,11} + \sigma_{22,22}), \tag{8.45}$$

where all terms containing ν cancel. Hence,

$$\sigma_{11,11} + \sigma_{11,22} + \sigma_{22,11} + \sigma_{22,22} = 0 \tag{8.46}$$

or

$$\nabla^2(\sigma_{11} + \sigma_{22}) = 0. \tag{8.47}$$

This is the stress compatibility equation for two-dimensional isotropic elasticity with no body force. This equation is valid for both plane stress and plane strain, since material constants have disappeared, and there is thus no need to use the formal equivalence between plane stress and plane strain.

8.5 Airy Stress Function

The two-dimensional equilibrium equations are automatically satisfied if a scalar function $U(x_1, x_2)$, called the *Airy stress function*, is chosen such that

$$\sigma_{11} = U_{,22}, \quad \sigma_{22} = U_{,11}, \quad \sigma_{12} = -U_{,12}, \tag{8.48}$$

since

$$\begin{cases} \sigma_{11,1} + \sigma_{12,2} = U_{,221} - U_{,122} = 0, \\ \sigma_{21,1} + \sigma_{22,2} = -U_{,121} + U_{,112} = 0. \end{cases} \tag{8.49}$$

The two-dimensional stress compatibility equation, Eq. 8.47, implies

$$0 = \nabla^2(U_{,22} + U_{,11}) = U_{,2211} + U_{,2222} + U_{,1111} + U_{,1122} = U_{,1111} + 2U_{,1122} + U_{,2222}. \tag{8.50}$$

This last result can be written in the form

$$0 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) U = \nabla^2\nabla^2 U = \nabla^4 U. \tag{8.51}$$

That is, the Airy stress function for two-dimensional elasticity satisfies the biharmonic equation

$$\nabla^4 U = 0. \tag{8.52}$$

Note that, for both plane stress and plane strain, the Airy stress function satisfies the same differential equation. The only difference is that, after the stress components are found, the strain components are found by different relations (derived in §8.3).

There are two ways in which the biharmonic equation can be used to solve elasticity problems:

1. Pick functions (such as polynomials of various degrees) with suitably chosen coefficients, and see what problem was solved.

2. Pose a problem of interest, and attempt a solution using Airy's stress function.

Both approaches will be illustrated by example in subsequent sections.

8.6 Polynomial Solutions of the Biharmonic Equation

Let U be given by the second degree polynomial

$$U = \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2, \tag{8.53}$$

where a, b, and c are constants. The biharmonic equation is satisfied for all values of a, b, and c. The stresses are given by Eq. 8.48 as

$$\begin{cases} \sigma_{xx} = U_{,yy} = c, \\ \sigma_{yy} = U_{,xx} = a, \\ \sigma_{xy} = -U_{,xy} = -b. \end{cases} \tag{8.54}$$

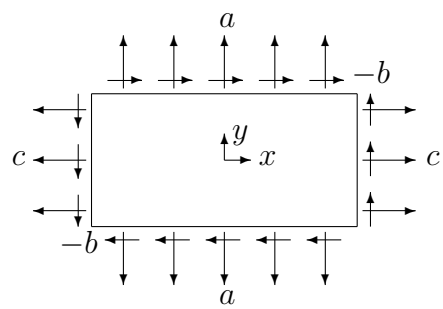


Figure 43: Stress Field for Second Degree Polynomial Stress Function.

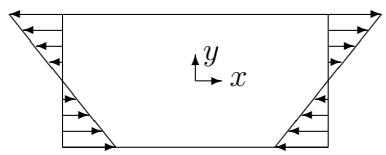


Figure 44: Pure Bending from Third Degree Polynomial Stress Function.

That is, all stress components are constant. For a rectangular strip, the stresses are shown in Fig. 43. This stress function thus represents a combination of uniform tensions or compressions in two directions and a uniform shear.

We now consider U given by the third degree polynomial

$$U = \frac{1}{6}ax^3 + \frac{1}{2}bx^2y + \frac{1}{2}cxy^2 + \frac{1}{6}dy^3. \tag{8.55}$$

Since

$$\nabla^4 U = U_{,1111} + 2U_{,1122} + U_{,2222}, \tag{8.56}$$

this third degree polynomial satisfies the biharmonic equation for all values of a, b, c, and d. The stresses are given by Eq. 8.48 as

$$\begin{cases} \sigma_{xx} = U_{,yy} = cx + dy, \\ \sigma_{yy} = U_{,xx} = ax + by, \\ \sigma_{xy} = -U_{,xy} = -bx - cy. \end{cases} \tag{8.57}$$

Several problems have been solved depending on the constants selected. If $a = b = c = 0$ and $d \neq 0$, the stress field corresponds to pure bending, as shown in Fig. 44. If $a = c = d = 0$ and $b \neq 0$, we have the stress field shown in Fig. 45. The symmetry of the tractions shown in that figure results from having the coordinate origin at the center of the slab. Changing the location of the origin would change the stress field. For example, if the origin were located on the left side, there would be no shear stress on the left edge.

Consider U given by the fourth degree polynomial

$$U = \frac{1}{12}ax^4 + \frac{1}{6}bx^3y + \frac{1}{2}cx^2y^2 + \frac{1}{6}dxy^3 + \frac{1}{12}ey^4. \tag{8.58}$$

In this case,

$$\nabla^4 U = U_{,xxxx} + 2U_{,xxyy} + U_{,yyyy} = 2a + 4c + 2e = 0 \tag{8.59}$$

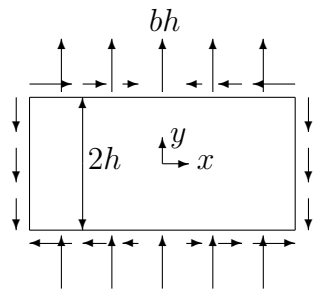


Figure 45: Stress Field from a Third Degree Polynomial Stress Function.

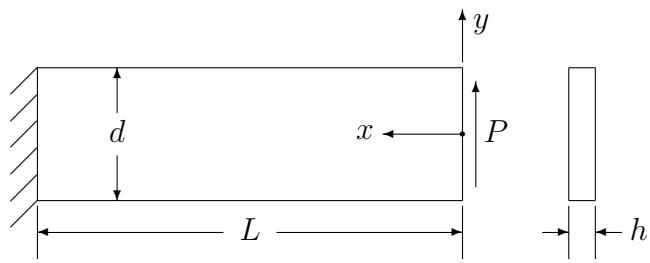


Figure 46: Cantilever Beam With End Load.

or

$$a + 2c + e = 0. \tag{8.60}$$

Thus, for fourth and higher degree polynomials, the coefficients cannot be picked arbitrarily.

8.7 Bending of Narrow Cantilever of Rectangular Cross-Section Under End Load

Consider the end-loaded cantilever beam of length L shown in Fig. 46. Let P denote the resultant load, whose precise distribution is left unspecified. If the thickness h is small compared with the beam depth d, we can assume plane stress (i.e., we assume $h \ll d$). We also assume zero tractions on the top and bottom edges. Since we seek the displacement and stress fields resulting from the applied load P, we ignore gravity. It is convenient to use a coordinate system whose origin is at the free end.

Since the bending moment in the beam is proportional to the distance x from the load, and, for bending, we would expect σ_{xx} to be proportional to y, we will attempt a solution of the form [22]

$$\sigma_{xx} = U_{,yy} = c_1 xy, \tag{8.61}$$

where U is the Airy stress function, and c_1 is a constant. This function can be integrated to yield

$$U_{,y} = \frac{1}{2} c_1 x y^2 + f_1(x), \tag{8.62}$$

where $f_1(x)$ is an unknown function of x. We integrate again to obtain

$$U = \frac{1}{6} c_1 x y^3 + y f_1(x) + f_2(x), \tag{8.63}$$

where $f_2(x)$ is another unknown function of x. Since U satisfies the biharmonic equation,

$$\nabla^4 U = U_{,1111} + 2U_{,1122} + U_{,2222} = y\frac{d^4 f_1}{dx^4} + \frac{d^4 f_2}{dx^4} = 0. \tag{8.64}$$

Since this equation must hold for all x and y, and only the first term depends on y, both terms must vanish. Thus,

$$\frac{d^4 f_1}{dx^4} = 0, \quad \frac{d^4 f_2}{dx^4} = 0 \tag{8.65}$$

and, hence,

$$f_1(x) = c_2 x^3 + c_3 x^2 + c_4 x + c_5, \tag{8.66}$$

$$f_2(x) = c_6 x^3 + c_7 x^2 + c_8 x + c_9. \tag{8.67}$$

Thus, from Eq. 8.63,

$$U = \frac{1}{6}c_1 x y^3 + y(c_2 x^3 + c_3 x^2 + c_4 x + c_5) + c_6 x^3 + c_7 x^2 + c_8 x + c_9, \tag{8.68}$$

where we will attempt to determine the nine constants from the various boundary conditions.

Given this stress function, it follows that

$$\begin{cases} \sigma_{xx} = U_{,yy} = c_1 x y, \\ \sigma_{yy} = U_{,xx} = 6(c_2 y + c_6)x + 2(c_3 y + c_7), \\ \sigma_{xy} = -U_{,xy} = -\frac{1}{2}c_1 y^2 - 3c_2 x^2 - 2c_3 x - c_4. \end{cases} \tag{8.69}$$

Since $\sigma_{yy} = 0$ on the top and bottom faces ($y = \pm d/2$),

$$6\left(c_2\frac{d}{2} + c_6\right)x + 2\left(c_3\frac{d}{2} + c_7\right) = 0, \tag{8.70}$$

$$6\left(-c_2\frac{d}{2} + c_6\right)x + 2\left(-c_3\frac{d}{2} + c_7\right) = 0, \tag{8.71}$$

which must hold for all x. Hence, each polynomial coefficient must vanish:

$$\begin{cases} c_2\frac{d}{2} + c_6 = 0, \\ -c_2\frac{d}{2} + c_6 = 0, \end{cases} \qquad \begin{cases} c_3\frac{d}{2} + c_7 = 0, \\ -c_3\frac{d}{2} + c_7 = 0. \end{cases} \tag{8.72}$$

These two pairs of simultaneous equations imply $c_2 = c_3 = c_6 = c_7 = 0$ and

$$\begin{cases} \sigma_{xx} = c_1 x y, \\ \sigma_{yy} = 0, \\ \sigma_{xy} = -\frac{1}{2}c_1 y^2 - c_4. \end{cases} \tag{8.73}$$

The boundary condition of zero shear on the top and bottom faces ($\sigma_{xy} = 0$ on $y = \pm d/2$) supplies one new equation and implies

$$c_4 = -\frac{1}{8}c_1 d^2. \tag{8.74}$$

Hence,

$$\begin{cases} \sigma_{xx} = c_1 xy, \\ \sigma_{yy} = 0, \\ \sigma_{xy} = -\frac{1}{8} c_1 (4y^2 - d^2). \end{cases} \tag{8.75}$$

The boundary condition that the resultant shear load at $x = 0$ is P implies

$$\int_{-d/2}^{d/2} \sigma_{xy}\, h\, dy = -P, \tag{8.76}$$

where the minus sign on the right-hand side follows from the sign convention for positive shear stress (Fig. 19, p. 35). The substitution of Eq. 8.75c into this equation yields

$$P = \frac{1}{8} c_1 h \int_{-d/2}^{d/2} (4y^2 - d^2)\, dy = -\frac{1}{12} c_1 h d^3. \tag{8.77}$$

Thus,

$$c_1 = \frac{-12P}{hd^3} = -\frac{P}{I}, \tag{8.78}$$

where I, the cross-section moment of inertia with respect to the z-axis, is

$$I = \frac{1}{12} h d^3. \tag{8.79}$$

Thus, Eq. 8.75 becomes

$$\begin{cases} \sigma_{xx} = -\dfrac{Pxy}{I}, \\ \sigma_{yy} = 0, \\ \sigma_{xy} = -\dfrac{P}{2I}\left(\dfrac{d^2}{4} - y^2\right). \end{cases} \tag{8.80}$$

We have several observations about this solution:

1. The solution agrees completely with the elementary mechanics of materials solution.

2. The shear stress varies parabolically in y at all locations x, including the loaded end at $x = 0$ and the fixed end at $x = L$. Thus, at $x = 0$, the shear stress boundary condition (the distribution of which was originally left unspecified) must also be specified to vary parabolically for Eq. 8.80 to be the solution. However, by Saint-Venant's principle, the solution would be OK at locations sufficiently far from the load. (Saint-Venant's principle states that the stress and strain fields resulting from statically equivalent systems of loads differ only in the immediate neighborhood of the loading. Stresses are essentially the same for statically equivalent loading systems at distances large in comparison with the dimensions of the surface over which the loads are applied.)

3. Since the shear stress at the fixed end must also vary parabolically, we have, in effect, solved a traction problem, not a mixed problem. (That is, specifying the stresses on the fixed end precludes our also specifying the displacement field there.) As a result, we would expect that the corresponding displacement field (to be derived below) would have three undetermined constants corresponding to the three degrees of freedom of rigid body motion possible in two-dimensional mechanics.

4. Constants c_5, c_8, and c_9 in U were not determined. However, since the stresses do not depend on these constants, they were not needed.

We now compute the displacement field corresponding to the stress field derived in Eq. 8.80. In general, we must integrate the strain-displacement equations. Let the displacement field be (u, v), in which case

$$\frac{\partial u}{\partial x} = \varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy} = -\frac{Pxy}{EI}, \tag{8.81}$$

$$\frac{\partial v}{\partial y} = \varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\sigma_{xx} = \frac{\nu Pxy}{EI}, \tag{8.82}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\varepsilon_{xy} = \frac{\sigma_{xy}}{G} = -\frac{(1+\nu)P}{EI}\left(\frac{d^2}{4} - y^2\right). \tag{8.83}$$

We can integrate the first two of these equations to obtain

$$u = -\frac{Px^2y}{2EI} + g_1(y), \tag{8.84}$$

$$v = \frac{\nu Pxy^2}{2EI} + g_2(x), \tag{8.85}$$

where $g_1(y)$ and $g_2(x)$ are unknown functions. The substitution of these two equations into Eq. 8.83 yields

$$-\frac{Px^2}{2EI} + g_1'(y) + \frac{\nu Py^2}{2EI} + g_2'(x) = -\frac{(1+\nu)P}{EI}\left(\frac{d^2}{4} - y^2\right), \tag{8.86}$$

which can be regrouped into separate functions of x and y:

$$g_1'(y) - \frac{P}{EI}\left(1 + \frac{\nu}{2}\right)y^2 = -g_2'(x) + \frac{Px^2}{2EI} - \frac{(1+\nu)Pd^2}{4EI}. \tag{8.87}$$

Since this equation, which equates a function of y with a function of x, must hold for all x and y, each function must be a constant a_1. Thus,

$$g_1'(y) = \frac{P}{EI}\left(1 + \frac{\nu}{2}\right)y^2 + a_1, \tag{8.88}$$

$$g_2'(x) = \frac{Px^2}{2EI} - \frac{(1+\nu)Pd^2}{4EI} - a_1. \tag{8.89}$$

These two equations can be integrated to yield

$$g_1(y) = \frac{P}{3EI}\left(1 + \frac{\nu}{2}\right)y^3 + a_1 y + a_2, \tag{8.90}$$

$$g_2(x) = \frac{Px^3}{6EI} - \frac{(1+\nu)Pd^2x}{4EI} - a_1 x + a_3. \tag{8.91}$$

Then, from Eqs. 8.84 and 8.85,

$$u = -\frac{P}{2EI}x^2y + \frac{P}{3EI}\left(1+\frac{\nu}{2}\right)y^3 + a_1y + a_2, \tag{8.92}$$

$$v = \frac{\nu P}{2EI}xy^2 + \frac{P}{6EI}x^3 - \frac{(1+\nu)Pd^2}{4EI}x - a_1x + a_3. \tag{8.93}$$

Note that this displacement solution has three undetermined constants of integration. As mentioned in the discussion following Eq. 8.80, these three constants correspond to the three degrees of freedom of rigid body motion (x translation, y translation, and rotation) possible in a two-dimensional traction problem. Thus, for uniqueness, we must specify three additional conditions sufficient to prevent such rigid body motion. Since the choice of constraints to prevent rigid body motion is not unique, the final displacement field will depend on that choice. Since we would like to compare our final solution with the elementary beam bending theory, we will assume that the point $(L, 0)$ is fixed and has zero slope:

$$u(L,0) = v(L,0) = \frac{\partial v(L,0)}{\partial x} = 0. \tag{8.94}$$

The first of these conditions implies $a_2 = 0$. The second condition, $v(L, 0) = 0$, implies

$$\frac{PL^3}{6EI} - \frac{(1+\nu)Pd^2L}{4EI} - a_1L + a_3 = 0. \tag{8.95}$$

To impose the third condition, we first compute the slope

$$\frac{\partial v}{\partial x} = \frac{\nu Py^2}{2EI} + \frac{Px^2}{2EI} - \frac{(1+\nu)Pd^2}{4EI} - a_1, \tag{8.96}$$

so that zero slope at $(L, 0)$ implies

$$\frac{PL^2}{2EI} - \frac{(1+\nu)Pd^2}{4EI} - a_1 = 0. \tag{8.97}$$

Hence,

$$a_1 = \frac{PL^2}{2EI} - \frac{(1+\nu)Pd^2}{4EI}. \tag{8.98}$$

Substituting this result into Eq. 8.95 yields

$$a_3 = -\frac{PL^3}{6EI} + \frac{(1+\nu)Pd^2L}{4EI} + \frac{PL^3}{2EI} - \frac{(1+\nu)Pd^2L}{4EI} = \frac{PL^3}{3EI}. \tag{8.99}$$

Thus, from Eqs. 8.92 and 8.93,

$$u = -\frac{P}{2EI}x^2y + \frac{P}{3EI}\left(1+\frac{\nu}{2}\right)y^3 + \frac{P}{2EI}\left[L^2 - (1+\nu)\frac{d^2}{2}\right]y \tag{8.100}$$

and

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{(1+\nu)Pd^2x}{4EI} - \left[\frac{PL^2}{2EI} - \frac{(1+\nu)Pd^2}{4EI}\right]x + \frac{PL^3}{3EI} \tag{8.101}$$

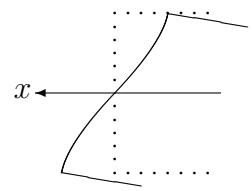


Figure 47: Distortion at Fixed End.

or

$$v = \frac{\nu P x y^2}{2EI} + \frac{Px^3}{6EI} - \frac{PL^2 x}{2EI} + \frac{PL^3}{3EI}. \tag{8.102}$$

To compare this elasticity solution to the beam theory solution, we compute the deflection curve (the deflection of the "neutral" axis $y = 0$),

$$v(x, 0) = \frac{Px^3}{6EI} - \frac{PL^2 x}{2EI} + \frac{PL^3}{3EI}, \tag{8.103}$$

from which it follows that

$$\frac{\partial^2 v}{\partial x^2} = \frac{Px}{EI} = \frac{M}{EI}, \tag{8.104}$$

where $M = Px$ is the bending moment at x. This is the Bernoulli-Euler formula in the theory of beam bending.

It is of interest to determine the distortion of the cross-section at the "fixed" end ($x = L$) due to the shearing stresses. From Eq. 8.100,

$$u(L, y) = \frac{P}{3EI}\left(1 + \frac{\nu}{2}\right) y^3 - \frac{P(1+\nu)d^2}{4EI} y, \tag{8.105}$$

and

$$\frac{\partial u(L, y)}{\partial y} = \frac{P}{EI}\left(1 + \frac{\nu}{2}\right) y^2 - \frac{P(1+\nu)d^2}{4EI}. \tag{8.106}$$

On the neutral axis at the fixed end,

$$\frac{\partial u(L, 0)}{\partial y} = -\frac{P(1+\nu)d^2}{4EI} = -\frac{Pd^2}{8GI} = -\frac{12Pd^2}{8Ghd^3} = -\frac{3P}{2GA}, \tag{8.107}$$

where A is the cross-sectional area. The distortion of the fixed end is sketched in Fig. 47. Thus, the displacement solution, Eqs. 8.100 and 8.102, satisfies the desired fixed boundary condition at $x = L$ only approximately. For slender beams ($d \ll L$), $u(L, y) = O(d^2 y)$, where we use the "big O" notation to represent terms of order $d^2 y$ as $d^2 y \to 0$. Thus, although $u(L, y) \neq 0$ for slender beams,

$$u(L, y) \ll u(0, y) \approx \frac{PL^2 y}{2EI} = O(L^2 y). \tag{8.108}$$

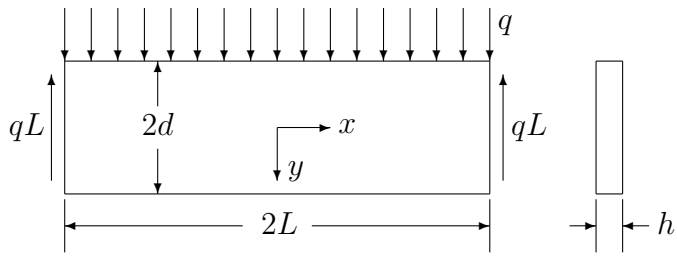


Figure 48: Uniformly-Loaded Beam Supported at Ends.

8.8 Bending of a Beam by Uniform Load

Consider the uniformly-loaded beam of length $2L$ and depth $2d$ shown in Fig. 48[20]. Let q denote the force per unit length applied on the top surface. Hence, the resultant reactions at the two ends would be qL. If the thickness h is small compared with the beam depth $2d$ (i.e., $h \ll d$), we can assume plane stress. The objective is to compute the stress field in the beam resulting from these loads. We ignore gravity.

The following boundary conditions are imposed:

$$\sigma_{xy}(x, \pm d) = 0, \tag{8.109}$$

$$\sigma_{yy}(x, d) = 0, \tag{8.110}$$

$$\sigma_{yy}(x, -d) = -\frac{q}{h}, \tag{8.111}$$

$$\int_{-d}^{d} \sigma_{xy}(\pm L, y)\, dy = \mp \frac{qL}{h}, \tag{8.112}$$

$$\int_{-d}^{d} \sigma_{xx}(\pm L, y)\, dy = 0, \tag{8.113}$$

$$\int_{-d}^{d} y\sigma_{xx}(\pm L, y)\, dy = 0. \tag{8.114}$$

The last two conditions indicate that there is no net longitudinal force or moment at the ends of the beam. The precise distribution of normal stress at the ends is left unspecified. The zero moment condition of Eq. 8.114 corresponds to a simple support.

Since the bending moment in a uniformly loaded beam is one polynomial order higher than the bending moment in an end-loaded beam (§8.7), we will attempt a solution using an Airy function of polynomial degree 5:

$$\begin{aligned} U = &\frac{1}{20}a_5x^5 + \frac{1}{12}b_5x^4y + \frac{1}{6}c_5x^3y^2 + \frac{1}{6}d_5x^2y^3 + \frac{1}{12}e_5xy^4 + \frac{1}{20}f_5y^5 \\ &+ \frac{1}{12}a_4x^4 + \frac{1}{6}b_4x^3y + \frac{1}{2}c_4x^2y^2 + \frac{1}{6}d_4xy^3 + \frac{1}{12}e_4y^4 \\ &+ \frac{1}{6}a_3x^3 + \frac{1}{2}b_3x^2y + \frac{1}{2}c_3xy^2 + \frac{1}{6}d_3y^3 \\ &+ \frac{1}{2}a_2x^2 + b_2xy + \frac{1}{2}c_2y^2, \end{aligned} \tag{8.115}$$

where the subscript on each coefficient indicates the order of the term in which it appears. This 18-term polynomial would be a complete fifth-order polynomial in two variables except that the three linear and constant terms, which would not contribute to the stress field, are omitted.

We first require that this polynomial satisfy the biharmonic equation:

$$\nabla^4 U = U_{,xxxx} + 2U_{,xxyy} + U_{,yyyy} = 0, \tag{8.116}$$

which implies

$$0 = 6a_5x + 2b_5y + 4c_5x + 4d_5y + 2e_5x + 6f_5y + 2a_4 + 4c_4 + 2e_4 \tag{8.117}$$

$$= 2(3a_5 + 2c_5 + e_5)x + 2(b_5 + 2d_5 + 3f_5)y + 2(a_4 + 2c_4 + e_4). \tag{8.118}$$

Since this equation must be an identity for all x and y, each polynomial coefficient must vanish:

$$\begin{cases} 3a_5 + 2c_5 + e_5 = 0, \\ b_5 + 2d_5 + 3f_5 = 0, \\ a_4 + 2c_4 + e_4 = 0. \end{cases} \tag{8.119}$$

Given this stress function, the stresses are

$$\begin{cases} \sigma_{xx} = U_{,yy} = \frac{1}{3}c_5x^3 + d_5x^2y + e_5xy^2 + f_5y^3 + c_4x^2 + d_4xy + e_4y^2 + c_3x + d_3y + c_2, \\ \sigma_{yy} = U_{,xx} = a_5x^3 + b_5x^2y + c_5xy^2 + \frac{1}{3}d_5y^3 + a_4x^2 + b_4xy + c_4y^2 + a_3x + b_3y + a_2, \\ \sigma_{xy} = -U_{,xy} = -\frac{1}{3}b_5x^3 - c_5x^2y - d_5xy^2 - \frac{1}{3}e_5y^3 - \frac{1}{2}b_4x^2 - 2c_4xy - \frac{1}{2}d_4y^2 - b_3x - c_3y - b_2. \end{cases} \tag{8.120}$$

Notice that each of these three expressions is a complete cubic in two variables. The boundary conditions in Eqs. 8.109–8.114 can be used to evaluate the 18 unknown polynomial coefficents, of which 15 are independent, given Eq. 8.119. However, the effort required can be reduced considerably by taking account of the symmetry of the problem. We observe from Fig. 48 that, as a consequence of symmetry, both σ_{xx} and σ_{yy} must be even functions of x. In addition, since the shear stresses on the side supports are of opposite signs, σ_{xy} must be an odd function of x. Consequently,

$$a_5 = c_5 = e_5 = b_4 = d_4 = a_3 = c_3 = b_2 = 0, \tag{8.121}$$

and the stress field simplifies to

$$\sigma_{xx} = d_5x^2y + f_5y^3 + c_4x^2 + e_4y^2 + d_3y + c_2, \tag{8.122}$$

$$\sigma_{yy} = b_5x^2y + \frac{1}{3}d_5y^3 + a_4x^2 + c_4y^2 + b_3y + a_2, \tag{8.123}$$

$$\sigma_{xy} = -\frac{1}{3}b_5x^3 - d_5xy^2 - 2c_4xy - b_3x. \tag{8.124}$$

Notice that, as a result of Eq. 8.121, Eq. 8.119a is automatically satisfied, and the other two parts of Eqs. 8.119 are unaffected.

We now apply the various boundary conditions. Eqs. 8.109 imply

$$\begin{cases} -\sigma_{xy}(x, d) = \frac{1}{3}b_5x^3 + (d_5d^2 + 2c_4d + b_3)x = 0, \\ -\sigma_{xy}(x, -d) = \frac{1}{3}b_5x^3 + (d_5d^2 - 2c_4d + b_3)x = 0. \end{cases} \tag{8.125}$$

Since these two equations must hold for all x, it follows that

$$b_5 = c_4 = 0, \quad d_5 d^2 + b_3 = 0, \tag{8.126}$$

and the stress field further simplifies to

$$\sigma_{xx} = d_5 x^2 y + f_5 y^3 + e_4 y^2 + d_3 y + c_2, \tag{8.127}$$

$$\sigma_{yy} = \frac{1}{3} d_5 y^3 + a_4 x^2 - d_5 d^2 y + a_2, \tag{8.128}$$

$$\sigma_{xy} = -d_5 x y^2 + d_5 d^2 x = d_5 (d^2 - y^2) x. \tag{8.129}$$

Eq. 8.110 implies

$$\sigma_{yy}(x, d) = a_4 x^2 + \frac{1}{3} d_5 d^3 - d_5 d^3 + a_2 = 0, \tag{8.130}$$

which must hold for all x. Hence,

$$a_4 = 0, \quad a_2 = \frac{2}{3} d_5 d^3. \tag{8.131}$$

Similarly, Eq. 8.111 and these last two results imply

$$\sigma_{yy}(x, -d) = -\frac{1}{3} d_5 d^3 + d_5 d^3 + \frac{2}{3} d_5 d^3 = -\frac{q}{h} \tag{8.132}$$

or

$$d_5 = -\frac{3q}{4hd^3} = -\frac{q}{2I}, \tag{8.133}$$

where I is the moment of inertia of the cross-sectional area with respect to the z-axis:

$$I = \frac{2}{3} h d^3. \tag{8.134}$$

Also, from Eqs. 8.119, 8.126, and 8.131,

$$e_4 = 0, \quad f_5 = -\frac{2}{3} d_5 = \frac{q}{3I}, \quad a_2 = -\frac{qd^3}{3I}, \tag{8.135}$$

and the stress field further simplifies to

$$\sigma_{xx} = -\frac{q}{2I} x^2 y + \frac{q}{3I} y^3 + d_3 y + c_2, \tag{8.136}$$

$$\sigma_{yy} = -\frac{q}{6I} y^3 + \frac{qd^2}{2I} y - \frac{qd^3}{3I}, \tag{8.137}$$

$$\sigma_{xy} = -\frac{q}{2I} (d^2 - y^2) x. \tag{8.138}$$

Two constants (d_3 and c_2) remain to be evaluated.

Eq. 8.112 is automatically satisfied. To impose Eq. 8.113a, we note that the terms in the integrand $\sigma_{xx}(L, y)$ which are odd in y do not contribute to the integral. Hence, this boundary condition implies $c_2 = 0$. Eq. 8.113b is then automatically satisfied.

The last boundary condition, Eq. 8.114, implies

$$-\frac{qL^2}{2I}\cdot\frac{d^3}{3}+\frac{q}{3I}\cdot\frac{d^5}{5}+d_3\frac{d^3}{3}=0 \tag{8.139}$$

or

$$d_3=\frac{q}{I}\left(\frac{L^2}{2}-\frac{d^2}{5}\right). \tag{8.140}$$

Thus, with all constants evaluated, the final solution for the stress field is

$$\sigma_{xx}=\frac{q}{2I}(L^2-x^2)y+\frac{q}{I}\left(\frac{1}{3}y^3-\frac{1}{5}d^2y\right), \tag{8.141}$$

$$\sigma_{yy}=-\frac{q}{6I}y^3+\frac{qd^2}{2I}y-\frac{qd^3}{3I}, \tag{8.142}$$

$$\sigma_{xy}=-\frac{q}{2I}(d^2-y^2)x. \tag{8.143}$$

It is of interest to compare this solution with that derived in elementary beam theory. The first term in Eq. 8.141 represents the normal stresses due to bending derived in elementary beam theory; the second term can thus be thought of as a "correction" to the elementary theory. This correction is independent of x and, if $d \ll L$, small compared with the maximum stress σ_{xx}. Thus, the elementary theory is a good approximation for long slender beams. We also note that the normal stresses at the ends have a resultant force and resultant moment equal to zero. Hence, from Saint-Venant's principle, their effect away from the ends will be small. The elementary theory also ignores the normal stress component σ_{yy}, which must be nonzero on the top surface. Eq. 8.142 shows that there are compressive stresses σ_{yy} between the longitudinal fibers of the beam. The parabolic distribution of shear stress derived in Eq. 8.143 corresponds to that derived in the elementary theory.

9 General Theorems of Infinitesimal Elastostatics

9.1 Work Theorem

A stress field σ_{ij} is defined as *statically admissible* if it satisfies the equations of static equilibrium and the traction boundary conditions:

$$\begin{cases} \sigma_{ij,j}+\rho f_i=0, \\ \sigma_{ij}n_j=T_i, \end{cases} \tag{9.1}$$

where the upper case T_i is used to denote the stress vector on the *boundary* of a body. One implication of static admissibility is that the stress field must be sufficiently smooth so that the indicated derivatives can be taken.

A strain field $\bar{\varepsilon}_{ij}$ is defined as *kinematically admissible* if it satisfies the strain-displacement equations and the displacement boundary conditions:

$$\begin{cases} \bar{\varepsilon}_{ij}=\frac{1}{2}(\bar{u}_{i,j}+\bar{u}_{j,i}), \\ \bar{u}_i=\bar{U}_i, \end{cases} \tag{9.2}$$

where $\bar{U}_i$ denotes the boundary displacements. One implication of kinematic admissibility is that the displacement field must be sufficiently smooth so that the indicated derivatives can be taken.

In the above two definitions, the bar is used over the displacements and strains to emphasize that the stress and strain fields do not have to be related to each other (i.e., they do not have to be associated with the same problem).

For a given body, we now consider the work done by the surface tractions for one problem acting through the boundary displacements associated with another problem:

$$\oint_S T_i \bar{u}_i \, dS = \oint_S \sigma_{ij} n_j \bar{u}_i \, dS = \int_V (\sigma_{ij}\bar{u}_i)_{,j} \, dV = \int_V (\sigma_{ij,j}\bar{u}_i + \sigma_{ij}\bar{u}_{i,j}) \, dV$$
$$= \int_V (-\rho f_i \bar{u}_i + \sigma_{ij}\bar{\varepsilon}_{ij}) \, dV. \tag{9.3}$$

Thus, from the first and last expressions,

$$\oint_S T_i \bar{u}_i \, dS + \int_V \rho f_i \bar{u}_i \, dV = \int_V \sigma_{ij}\bar{\varepsilon}_{ij} \, dV. \tag{9.4}$$

That is, the work of the surface tractions and body forces associated with one problem (for a given body) acting through the displacements associated with another problem (for the same body) is equal to a "strain energy" computed using the stress field of the first problem and the strain field of the second problem. This result is known as the *work theorem*.

So far we have proved that Eqs. 9.1 and 9.2 imply Eq. 9.4. It can also be shown that Eqs. 9.4 and 9.2 imply Eq. 9.1. That is, given σ_{ij}, f_i, and T_i, if the work theorem is satisfied for *every* kinematically admissible strain-displacement field, then σ_{ij} is also statically admissible. To prove this assertion, we observe that the right-hand side of the work theorem is given by

$$\int_V \sigma_{ij}\bar{\varepsilon}_{ij} \, dV = \int_V \sigma_{ij}\bar{u}_{i,j} \, dV = \int_V [(\sigma_{ij}\bar{u}_i)_{,j} - \sigma_{ij,j}\bar{u}_i] \, dV = \oint_S \sigma_{ij} n_j \bar{u}_i \, dS - \int_V \sigma_{ij,j}\bar{u}_i \, dV. \tag{9.5}$$

Thus, if we combine this last result with the left-hand side of the work theorem, we obtain

$$\oint_S T_i \bar{u}_i \, dS + \int_V \rho f_i \bar{u}_i \, dV = \oint_S \sigma_{ij} n_j \bar{u}_i \, dS - \int_V \sigma_{ij,j}\bar{u}_i \, dV \tag{9.6}$$

or, with a re-arrangement,

$$\int_V (\sigma_{ij,j} + \rho f_i)\bar{u}_i \, dV = \oint_S (\sigma_{ij} n_j - T_i)\bar{u}_i \, dS. \tag{9.7}$$

If the work theorem holds for every kinematically admissible strain-displacement field, it holds, in particular, for a field with zero surface displacements, in which case

$$\int_V (\sigma_{ij,j} + \rho f_i)\bar{u}_i \, dV = 0, \tag{9.8}$$

which must hold for all internal distributions of $\bar{u}_i$. Thus,

$$\sigma_{ij,j} + \rho f_i = 0 \quad \text{in } V. \tag{9.9}$$

Also, if we substitute this result into Eq. 9.7, we obtain

$$\oint_S (\sigma_{ij} n_j - T_i)\bar{u}_i \, dS = 0. \tag{9.10}$$

At portions of the boundary where $\bar{u}_i$ is *not* specified, this equation must hold for arbitrary $\bar{u}_i$, which implies that the integrand vanishes and

$$T_i = \sigma_{ij} n_j. \tag{9.11}$$

However, on portions of the boundary where $\bar{u}_i$ *is* specified, we cannot conclude (from the above equations) that Eq. 9.11 holds.

9.2 Betti's Reciprocal Theorem

Assume that we have two different elasticity solutions for the same body. That is, assume we know, for some problem,

$$\sigma_{ij}, \quad \rho f_i, \quad T_i, \quad u_i, \quad \varepsilon_{ij},$$

where these field quantities satisfy all the equations of elasticity. If we have a second solution of a different elasticity problem for the same body, we would know

$$\bar{\sigma}_{ij}, \quad \rho \bar{f}_i, \quad \bar{T}_i, \quad \bar{u}_i, \quad \bar{\varepsilon}_{ij},$$

which also satisfy all the equations of elasticity. Betti's theorem states that, if an elastic body is subjected to two systems of body and surface forces, the work that would be done by the first system of forces acting through the displacements of the second system is equal to the work that would be done by the forces of the second system acting through the displacements of the first system. That is,

$$\int_V \rho f_i \bar{u}_i \, dV + \oint_S T_i \bar{u}_i \, dS = \int_V \rho \bar{f}_i u_i \, dV + \oint_S \bar{T}_i u_i \, dS. \tag{9.12}$$

To prove Betti's theorem, we observe that, according to the work theorem, Eq. 9.4, the left-hand side of Eq. 9.12, is given by

$$\int_V \rho f_i \bar{u}_i \, dV + \oint_S T_i \bar{u}_i \, dS = \int_V \sigma_{ij} \bar{\varepsilon}_{ij} \, dV. \tag{9.13}$$

Similarly, by an interchange of the two solutions, the right-hand side Eq. 9.12 is given by

$$\int_V \rho \bar{f}_i u_i \, dV + \oint_S \bar{T}_i u_i \, dS = \int_V \bar{\sigma}_{ij} \varepsilon_{ij} \, dV. \tag{9.14}$$

However, the integrands (and, hence, the integrals) in the right-hand sides of the last two equations are equal, since

$$\sigma_{ij}\bar{\varepsilon}_{ij} = c_{ijkl}\bar{\varepsilon}_{ij}\varepsilon_{kl} = c_{klij}\bar{\varepsilon}_{ij}\varepsilon_{kl} = c_{ijkl}\bar{\varepsilon}_{kl}\varepsilon_{ij} = \bar{\sigma}_{ij}\varepsilon_{ij}, \tag{9.15}$$

where the second equality follows from the symmetry $c_{ijkl} = c_{klij}$, and the third equality follows from the interchange of subscripts ij with kl. Betti's theorem is therefore proved.

9.3 Variational Principles

We consider first an example involving the system of linear algebraic equations

$$\mathbf{M}\mathbf{x} = \mathbf{y}, \tag{9.16}$$

where $\mathbf{M}$ is a square matrix, and $\mathbf{x}$ and $\mathbf{y}$ are vectors. Define the scalar function

$$F(\mathbf{x}) = \mathbf{x} \cdot \mathbf{M}\mathbf{x} - 2\mathbf{x} \cdot \mathbf{y}. \tag{9.17}$$

To determine the vector $\mathbf{x}$ which minimizes F (for fixed $\mathbf{M}$ and $\mathbf{y}$), we compute and set to zero the *first variation* (or, simply, the *variation*) of F:

$$0 = \delta F(\mathbf{x}) = \delta\mathbf{x} \cdot \mathbf{M}\mathbf{x} + \mathbf{x} \cdot \mathbf{M}\delta\mathbf{x} - 2\delta\mathbf{x} \cdot \mathbf{y} \tag{9.18}$$

$$= \delta\mathbf{x} \cdot \mathbf{M}\mathbf{x} + \delta\mathbf{x} \cdot \mathbf{M}^T\mathbf{x} - 2\delta\mathbf{x} \cdot \mathbf{y} \tag{9.19}$$

$$= 2\delta\mathbf{x} \cdot \left[\frac{1}{2}(\mathbf{M} + \mathbf{M}^T)\mathbf{x} - y\right]. \tag{9.20}$$

Since this last expression must vanish for all $\delta\mathbf{x}$, it follows that

$$\frac{1}{2}(\mathbf{M} + \mathbf{M}^T)\mathbf{x} = y. \tag{9.21}$$

That is, the vector $\mathbf{x}$ which minimizes the scalar function $F(\mathbf{x})$ is determined by Eq. 9.21. If $\mathbf{M}$ is symmetric, then Eq. 9.21 reduces to Eq. 9.16, the original system of equations. Thus, we have found that the vector $\mathbf{x}$ which mimimizes the functional, Eq. 9.17, is the solution of Eq. 9.16. In general, a *functional* is a function of a function.

We now wish to apply these ideas to elasticity. Recall that the total strain energy for a body is given by

$$W[\mathbf{u}] = \frac{1}{2}\int_V c_{ijkl} u_{i,j} u_{k,l}\, dV, \tag{9.22}$$

where we consider the strain energy to be a functional depending on the displacement vector $\mathbf{u}$. Admissible displacement fields are those such that $u_i = U_i(\mathbf{x})$ on the surface S of the body. Suppose $\bar{\mathbf{u}}$ is the minimizing admissible displacement field. Consider also a slightly different displacement field

$$\bar{\mathbf{u}}(\mathbf{x}) + \epsilon\boldsymbol{\eta}(\mathbf{x}),$$

where, to be admissible,

$$\epsilon\boldsymbol{\eta}(\mathbf{x}) = \mathbf{0} \quad \text{on } S. \tag{9.23}$$

Then, for any $\epsilon \neq 0$,

$$f(\epsilon) = W[\bar{\mathbf{u}}(\mathbf{x}) + \epsilon\boldsymbol{\eta}(\mathbf{x})] > W[\bar{\mathbf{u}}(\mathbf{x})]. \tag{9.24}$$

From Eq. 9.22,

$$f(\epsilon) = W[\bar{\mathbf{u}} + \epsilon\boldsymbol{\eta}] = W[\bar{\mathbf{u}}] + \epsilon Q[\bar{\mathbf{u}}, \boldsymbol{\eta}] + \epsilon^2 W[\boldsymbol{\eta}], \tag{9.25}$$

where

$$Q[\bar{\mathbf{u}}, \boldsymbol{\eta}] = \frac{1}{2}\int_V [c_{ijkl}\bar{u}_{i,j}\eta_{k,l} + c_{ijkl}\eta_{i,j}\bar{u}_{k,l}]\, dV = \int_V c_{ijkl}\eta_{i,j}\bar{u}_{k,l}\, dV. \tag{9.26}$$

The last equation follows from the symmetry $c_{ijkl} = c_{klij}$. Since $\bar{\mathbf{u}}$ is the minimizing admissible displacement field for $W[\mathbf{u}]$, $f(\epsilon)$ achieves its minimum for $\epsilon = 0$. That is, $f'(0) = 0$. Since

$$f'(\epsilon) = Q[\bar{\mathbf{u}}, \boldsymbol{\eta}] + 2\epsilon W[\boldsymbol{\eta}], \tag{9.27}$$

$f'(0) = 0$ implies

$$Q[\bar{\mathbf{u}}, \boldsymbol{\eta}] = 0. \tag{9.28}$$

Also, in order for the stationary point at $\epsilon = 0$ to be a minimum, the second derivative $f''(\epsilon) = 2W[\boldsymbol{\eta}]$ must be positive there, a condition which is automatically satisfied for the strain energy functional W.

Eq. 9.28 implies

$$0 = Q[\bar{\mathbf{u}}, \boldsymbol{\eta}] = \int_V c_{ijkl}\eta_{i,j}\bar{u}_{k,l}\, dV = \int_V \bar{\sigma}_{ij}\eta_{i,j}\, dV. \tag{9.29}$$

This equation, which must hold for all $\boldsymbol{\eta}$, does not imply that $\bar{\sigma}_{ij} = 0$, since we note that, if $\bar{\sigma}_{ij} = 1$,

$$\int_V \eta_{i,j}\, dV = \oint_S \eta_i n_j\, dS = 0, \tag{9.30}$$

since $\boldsymbol{\eta}$ must vanish on the surface (Eq. 9.23). However, from Eq. 9.29,

$$\begin{aligned} 0 &= \int_V \bar{\sigma}_{ij}\eta_{i,j}\, dV = \int_V [(\bar{\sigma}_{ij}\eta_i)_{,j} - \bar{\sigma}_{ij,j}\eta_i]\, dV \\ &= \oint_S \bar{\sigma}_{ij}\eta_i n_j\, dS - \int_V \bar{\sigma}_{ij,j}\eta_i\, dV = -\int_V \bar{\sigma}_{ij,j}\eta_i\, dV, \end{aligned} \tag{9.31}$$

where $\boldsymbol{\eta}$ vanishes on the surface. Since this last expression must vanish for all $\boldsymbol{\eta}$, it follows that $\bar{\sigma}_{ij,j} = 0$ pointwise at interior points in the volume V. That is, we have proved that the stress field which corresponds to the minimizing displacement field $\bar{u}_i$ satisfies the equilibrium equation

$$\bar{\sigma}_{ij,j} = (c_{ijkl}\bar{u}_{k,l})_{,j} = 0. \tag{9.32}$$

Alternatively, given the displacements on the boundary, the corresponding stress field is that which minimizes the total strain energy.

9.4 Theorem of Minimum Potential Energy

In the preceding section, we considered problems for which there were no body forces, and the displacements were prescribed over the entire surface. We now consider more general problems which have body forces and for which the boundary conditions involve both displacements (specified on S_u) and surface tractions (specified on S_t) (Fig. 49). An admissible displacement field is one which satisfies the boundary condition $\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x})$ on S_u and is smooth enough to be differentiated to yield strains. We make the following assumptions:

1. $c_{ijkl} = c_{klij}$,

2. $c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > 0$ if $\boldsymbol{\varepsilon} \neq \mathbf{0}$, and

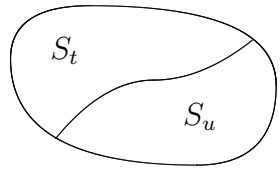


Figure 49: Surfaces S_u and S_t.

3. S_u is not merely a colinear set of points.

According to the *theorem of minimum potential energy*, for admissible $\mathbf{u}$, if the functional

$$W[\mathbf{u}] = \frac{1}{2}\int_V c_{ijkl}u_{i,j}u_{k,l}\,dV - \int_V \rho f_i u_i\,dV - \int_{S_t} T_i u_i\,dS \tag{9.33}$$

is stationary at $\mathbf{u} = \bar{\mathbf{u}}$, then

$$\begin{cases} (c_{ijkl}\bar{u}_{k,l})_{,j} + \rho f_i = 0, \\ \bar{u}_i = U_i \ \text{ on } S_u, \\ \bar{\sigma}_{ij}n_j = c_{ijkl}\bar{u}_{k,l}n_j = T_i \ \text{ on } S_t, \\ W[\bar{\mathbf{u}}] < W[\mathbf{u}] \ \text{ for } \mathbf{u} \neq \bar{\mathbf{u}}. \end{cases} \tag{9.34}$$

The potential energy W consists of the sum of the strain energy and the potential of the applied loads. In words, we can state the Minimum Potential Energy theorem as follows: Of all kinematically admissible displacement fields (those which satisfy the prescribed displacement boundary conditions), the potential energy W achieves its absolute minimum for the displacement field which is that of the equilibrium state. An equilibrium displacement field is one for which the stresses obtained from Hooke's law satisfy both equilibrium and the prescribed traction boundary conditions.

To prove this theorem, we first compute and set to zero the first variation of the functional W for fixed f_i, T_i, and U_i:

$$\begin{aligned} 0 &= \delta W[\mathbf{u}] && (9.35)\\ &= \int_V \frac{1}{2}[c_{ijkl}\delta u_{i,j}u_{k,l} + c_{ijkl}u_{i,j}\delta u_{k,l}]\,dV - \int_V \rho f_i \delta u_i\,dV - \int_{S_t} T_i\delta u_i\,dS && (9.36)\\ &= \int_V \delta u_{i,j}c_{ijkl}u_{k,l}\,dV - \int_V \rho f_i\delta u_i\,dV - \int_{S_t} T_i\delta u_i\,dS && (9.37)\\ &= \int_V [(\delta u_i c_{ijkl}u_{k,l})_{,j} - \delta u_i(c_{ijkl}u_{k,l})_{,j}]\,dV - \int_V \rho f_i\delta u_i\,dV - \int_{S_t} T_i\delta u_i\,dS && (9.38)\\ &= \oint_S \delta u_i c_{ijkl}u_{k,l}n_j\,dS - \int_V \delta u_i[(c_{ijkl}u_{k,l})_{,j} + \rho f_i]\,dV - \int_{S_t} T_i\delta u_i\,dS. && (9.39) \end{aligned}$$

Since $\mathbf{u} = \mathbf{U}$ on S_u, $\mathbf{u}$ is not allowed to vary on that portion of the boundary (i.e., $\delta\mathbf{u} = \mathbf{0}$ on S_u). Hence, the first integral in this last equation has contributions only on S_t, and

$$0 = \delta W[\mathbf{u}] = \int_{S_t} \delta u_i(c_{ijkl}u_{k,l}n_j - T_i)\,dS - \int_V \delta u_i[(c_{ijkl}u_{k,l})_{,j} + \rho f_i]\,dV. \tag{9.40}$$

Since this equation must hold for all $\delta\mathbf{u}$, it follows that

$$\begin{cases} (c_{ijkl}u_{k,l})_{,j} + \rho f_i = 0 \ \text{ in } V, \\ c_{ijkl}u_{k,l}n_j = \sigma_{ij}n_j = T_i \ \text{ on } S_t, \end{cases} \tag{9.41}$$

and the first part of the minimum potential energy theorem is proved.

To prove the remaining part of the theorem, Eq. 9.34d, we need to show that the stationary value of the potential energy W is a minimum (although one might expect from physical intuition that there can be no *maximum* energy for admissible displacement fields). We use an approach similar to that used in the preceding section, where we considered a displacement field

$$\bar{\mathbf{u}}(\mathbf{x}) + \epsilon\boldsymbol{\eta}(\mathbf{x}).$$

To be admissible,

$$\boldsymbol{\eta}(\mathbf{x}) = \mathbf{0} \quad \text{on } S_u. \tag{9.42}$$

We define $f(\epsilon)$ as the energy functional evaluated at $\bar{\mathbf{u}} + \epsilon\boldsymbol{\eta}$:

$$f(\epsilon) = W[\bar{\mathbf{u}} + \epsilon\boldsymbol{\eta}] \tag{9.43}$$

$$= W[\bar{\mathbf{u}}] + \epsilon\left\{Q[\bar{\mathbf{u}}, \boldsymbol{\eta}] - \int_V \rho f_i\eta_i\,dV - \int_{S_t} T_i\eta_i\,dS\right\} + \epsilon^2\frac{1}{2}\int_V c_{ijkl}\eta_{i,j}\eta_{k,l}\,dV, \tag{9.44}$$

where Q is the same functional defined in the preceding section:

$$Q[\bar{\mathbf{u}}, \boldsymbol{\eta}] = \frac{1}{2}\int_V [c_{ijkl}\bar{u}_{i,j}\eta_{k,l} + c_{ijkl}\eta_{i,j}\bar{u}_{k,l}]\,dV = \int_V c_{ijkl}\eta_{i,j}\bar{u}_{k,l}\,dV. \tag{9.45}$$

The last equation follows from the symmetry $c_{ijkl} = c_{klij}$.

If the energy functional is to be minimized for $\mathbf{u} = \bar{\mathbf{u}}$, it follows that $f'(0) = 0$ and $f''(0) > 0$, where

$$f'(\epsilon) = Q[\bar{\mathbf{u}}, \boldsymbol{\eta}] - \int_V \rho f_i\eta_i\,dV - \int_{S_t} T_i\eta_i\,dS + \epsilon\int_V c_{ijkl}\eta_{i,j}\eta_{k,l}\,dV. \tag{9.46}$$

and

$$f''(\epsilon) = \int_V c_{ijkl}\eta_{i,j}\eta_{k,l}\,dV. \tag{9.47}$$

Since $\delta W[\mathbf{u}] = 0$ from the first part of the proof, the statement $f'(0) = 0$ is equivalent and does not need to be proved again. Nevertheless, it would be instructive to show the alternative proof. The condition $f'(0) = 0$ implies

$$0 = \int_V c_{ijkl}\eta_{i,j}\bar{u}_{k,l}\,dV - \int_V \rho f_i\eta_i\,dV - \int_{S_t} T_i\eta_i\,dS \tag{9.48}$$

$$= \int_V \bar{\sigma}_{ij}\eta_{i,j}\,dV - \int_V \rho f_i\eta_i\,dV - \int_{S_t} T_i\eta_i\,dS \tag{9.49}$$

$$= \int_V [(\bar{\sigma}_{ij}\eta_i)_{,j} - \bar{\sigma}_{ij,j}\eta_i]\,dV - \int_V \rho f_i\eta_i\,dV - \int_{S_t} T_i\eta_i\,dS \tag{9.50}$$

$$= \oint_S \bar{\sigma}_{ij}\eta_i n_j\,dS - \int_V \eta_i(\bar{\sigma}_{ij,j} + \rho f_i)\,dV - \int_{S_t} T_i\eta_i\,dS, \tag{9.51}$$

where, since $\boldsymbol{\eta} = \mathbf{0}$ on S_u (Eq. 9.42), the first surface integral has contributions only on S_t, and

$$0 = \int_{S_t} \eta_i(\bar{\sigma}_{ij} n_j - T_i)\, dS - \int_V \eta_i(\bar{\sigma}_{ij,j} + \rho f_i)\, dV, \tag{9.52}$$

which implies Eq. 9.41 as before.

Note that the second derivative $f''(\epsilon)$ (Eq. 9.47) is positive everywhere including $\epsilon = 0$. Thus, the minimum energy expressed (and now proved) in Eq. 9.34d is an *absolute* minimum, not just a local minimum. Alternatively, from Eq. 9.44, since the term proportional to ϵ is zero, we could write

$$W[\bar{\mathbf{u}} + \epsilon\boldsymbol{\eta}] = W[\bar{\mathbf{u}}] + \frac{1}{2}\epsilon^2 \int_V c_{ijkl}\eta_{i,j}\eta_{k,l}\, dV. \tag{9.53}$$

If we define the new variable

$$\hat{\eta}_{ij} = \frac{1}{2}(\eta_{i,j} + \eta_{j,i}), \tag{9.54}$$

we can write Eq. 9.53 as

$$W[\bar{\mathbf{u}} + \epsilon\boldsymbol{\eta}] = W[\bar{\mathbf{u}}] + \frac{1}{2}\epsilon^2 \int_V c_{ijkl}\hat{\eta}_{ij}\hat{\eta}_{kl}\, dV, \tag{9.55}$$

where, from the second basic assumption at the beginning of this section, the integral is positive unless $\hat{\eta}_{ij} \equiv 0$. Thus, $W[\bar{\mathbf{u}} + \epsilon\boldsymbol{\eta}] > W[\bar{\mathbf{u}}]$ unless $\boldsymbol{\eta} \equiv \mathbf{0}$.

Since the energy associated with any displacement field different from $\bar{\mathbf{u}}$ is larger than the energy associated with $\bar{\mathbf{u}}$, a corollary of the minimum potential energy theorem is that the minimizing displacement field $\bar{\mathbf{u}}$ is unique.

9.5 Minimum Complementary Energy

We recall that a stress field σ_{ij} is statically admissible if it satisfies the equations of static equilibrium and the traction boundary conditions,

$$\begin{cases} \sigma_{ij,j} + \rho f_i = 0, \\ \sigma_{ij} n_j = T_i, \end{cases} \tag{9.56}$$

where the stresses are related to the strains by generalized Hooke's law,

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}. \tag{9.57}$$

We also assume the existence of a strain energy density (which is positive for nonzero strains):

$$w = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > 0 \quad (\boldsymbol{\varepsilon} \neq \mathbf{0}). \tag{9.58}$$

For this discussion, it is convenient to write Hooke's law in its inverse form,

$$\varepsilon_{ij} = s_{ijkl}\sigma_{kl}, \tag{9.59}$$

in which case w can alternatively be written in terms of stress,

$$w = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}s_{ijkl}\sigma_{ij}\sigma_{kl} > 0 \quad (\boldsymbol{\sigma} \neq \mathbf{0}). \tag{9.60}$$

Note that, in the absence of thermal effects (heating), $\boldsymbol{\varepsilon} = \mathbf{0}$ if, and only if, $\boldsymbol{\sigma} = \mathbf{0}$. Thus, since the strain energy density is positive for nonzero strain fields, it is also positive for nonzero stress fields.

We now define the *complementary energy* $\widehat{W}$ as

$$\widehat{W} = \int_V \frac{1}{2} s_{ijkl}\sigma_{ij}\sigma_{kl}\, dV - \int_{S_u} U_i \sigma_{ij} n_j\, dS, \tag{9.61}$$

where $\mathbf{U}$ is the vector of prescribed displacements on S_u. $\widehat{W}$ consists of the sum of the complementary strain energy and the potential of the boundary forces acting through the prescribed boundary displacements.

The Minimum Complementary Energy theorem can be stated as follows: Of all statically admissible stress fields (those which satisfy equilibrium and the prescribed traction boundary conditions), the complementary energy $\widehat{W}$ achieves its absolute minimum for the stress field which is that of the equilibrium state. An equilibrium stress field is one for which the strains obtained from the inverse Hooke's law satisfy the compatibility equations (so that the strain-displacement equations can be integrated to yield displacements), and the resulting displacements satisfy the prescribed displacement boundary conditions.

We now prove this theorem. Let $\bar{\boldsymbol{\sigma}}$ denote the equilibrium stress field. Then, for another stress field,

$$\sigma_{ij} = \bar{\sigma}_{ij} + \Delta\sigma_{ij}, \tag{9.62}$$

we wish to prove that

$$\widehat{W}[\bar{\boldsymbol{\sigma}} + \Delta\boldsymbol{\sigma}] - \widehat{W}[\bar{\boldsymbol{\sigma}}] > 0, \tag{9.63}$$

where both $\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\sigma}}$ are statically admissible. Then,

$$\widehat{W}[\bar{\boldsymbol{\sigma}} + \Delta\boldsymbol{\sigma}] - \widehat{W}[\bar{\boldsymbol{\sigma}}] \tag{9.64}$$

$$= \int_V \frac{1}{2} s_{ijkl}[(\bar{\sigma}_{ij} + \Delta\sigma_{ij})(\bar{\sigma}_{kl} + \Delta\sigma_{kl}) - \bar{\sigma}_{ij}\bar{\sigma}_{kl}]\, dV - \int_{S_u} U_i \Delta\sigma_{ij} n_j\, dS \tag{9.65}$$

$$= \int_V \frac{1}{2} s_{ijkl}(\Delta\sigma_{ij}\bar{\sigma}_{kl} + \bar{\sigma}_{ij}\Delta\sigma_{kl})\, dV + \int_V \frac{1}{2} s_{ijkl}\Delta\sigma_{ij}\Delta\sigma_{kl}\, dV - \int_{S_u} U_i \Delta\sigma_{ij} n_j\, dS, \tag{9.66}$$

where the middle term is positive for nonzero $\Delta\boldsymbol{\sigma}$. Thus,

$$\widehat{W}[\bar{\boldsymbol{\sigma}} + \Delta\boldsymbol{\sigma}] - \widehat{W}[\bar{\boldsymbol{\sigma}}] > \int_V s_{ijkl}\Delta\sigma_{ij}\bar{\sigma}_{kl}\, dV - \int_{S_u} U_i \Delta\sigma_{ij} n_j\, dS \tag{9.67}$$

$$= \int_V \Delta\sigma_{ij}\bar{\varepsilon}_{ij}\, dV - \int_{S_u} U_i \Delta\sigma_{ij} n_j\, dS \tag{9.68}$$

$$= \int_V \Delta\sigma_{ij}\bar{u}_{i,j}\, dV - \int_{S_u} U_i \Delta\sigma_{ij} n_j\, dS \tag{9.69}$$

$$= \int_V (\Delta\sigma_{ij}\bar{u}_i)_{,j}\, dV - \int_V (\Delta\sigma_{ij})_{,j}\bar{u}_i\, dV - \int_{S_u} U_i \Delta\sigma_{ij} n_j\, dS. \tag{9.70}$$

Note that the middle term in this last expression vanishes, since, if $\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\sigma}}$ both satisfy equilibrium,

$$\begin{cases} \sigma_{ij,j} + \rho f_i = 0, \\ \bar{\sigma}_{ij,j} + \rho f_i = 0, \end{cases} \tag{9.71}$$

the difference stress field satisfies

$$(\Delta\sigma_{ij})_{,j} = 0. \tag{9.72}$$

Then, from Eq. 9.70,

$$\widehat{W}[\bar{\boldsymbol{\sigma}} + \Delta\boldsymbol{\sigma}] - \widehat{W}[\bar{\boldsymbol{\sigma}}] > \oint_S \Delta\sigma_{ij}\bar{u}_i n_j \, dS - \int_{S_u} U_i \Delta\sigma_{ij} n_j \, dS \tag{9.73}$$

$$= \int_{S_u} \Delta\sigma_{ij}\bar{u}_i n_j \, dS - \int_{S_u} U_i \Delta\sigma_{ij} n_j \, dS \tag{9.74}$$

$$= \int_{S_u} \Delta\sigma_{ij} U_i n_j \, dS - \int_{S_u} U_i \Delta\sigma_{ij} n_j \, dS = 0. \tag{9.75}$$

In the above equations, Eq. 9.74 follows since the difference stress vector must vanish on S_t, and only contributions on S_u remain. Eq. 9.75 follows since $\bar{u}_i = U_i$ on S_u (i.e., $\bar{u}_i$ is the equilibrium solution). The minimum complementary energy theorem is thus proved.

We mention in passing (and without elaboration) that the minimum complementary energy theorem can alternatively be posed as a constrained optimization problem, in which one wants to minimize

$$\widehat{W} = \int_V \frac{1}{2} s_{ijkl}\sigma_{ij}\sigma_{kl} \, dV - \int_{S_u} U_i \sigma_{ij} n_j \, dS \tag{9.76}$$

subject to the constraints

$$\begin{cases} \sigma_{ij,j} + \rho f_i = 0 \ \text{ in } V, \\ \sigma_{ij} n_j = T_i \ \text{ on } S_t. \end{cases} \tag{9.77}$$

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