Analytical Solution of

Partial Differential Equations

by

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Preface

These lecture notes are intended to supplement a one-semester graduate-level engineering course at The George Washington University in classical techniques for the solution of linear partial differential equations that arise in engineering and physics. Included are Laplace's equation, Poisson's equation, the heat equation, and the wave equation in Cartesian, cylindrical, and spherical geometries. The emphasis is on separation of variables, Sturm-Liouville theory, Fourier series, eigenfunction expansions, and Fourier transform techniques. The main prerequisite is a standard undergraduate calculus sequence including ordinary differential equations. However, much of what is needed involving the solution of ordinary differential equations is covered when the need arises. An elementary knowledge of complex variables is also assumed in the discussion of the Fourier integral theorem and the Fourier transforms. In general, the mix of topics and level of presentation are aimed at upper-level undergraduates and first-year graduate students in mechanical, aerospace, and civil engineering. The applications are drawn mostly from heat conduction (for parabolic and elliptic problems) and mechanics (for hyperbolic problems). These notes also include a brief introduction to finite difference solution methods.

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Contents

List of Figures

1 Review of Notation and Integral Theorems

1.1 Vectors

We denote vector quantities in print using boldface letters (e.g., $\mathbf{A}$) and, when hand-written, with an underline (e.g., $\underline{A}$). The basis vectors in Cartesian coordinates are denoted $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ in the x, y, and z directions, respectively. We prefer these symbols rather than $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ to allow generality for other coordinate systems and to avoid confusion with the use of k for thermal conductivity in the heat equation. Thus, in Cartesian coordinates,

$$\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z, \tag{1.1}$$

where A_x, A_y, and A_z are the Cartesian components of $\mathbf{A}$.

The *scalar* (or *dot*) *product* of two vectors $\mathbf{A}$ and $\mathbf{B}$ in Fig. 1 is thus

$$\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \tag{1.2}$$

It can also be shown that this expression is equivalent to

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta = AB\cos\theta, \tag{1.3}$$

where A is the length of the vector $\mathbf{A}$, and θ is the angle between the two vectors. The *vector* (or *cross*) *product* of two vectors $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|(\sin\theta)\mathbf{e}_n = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1.4}$$

$$= (A_yB_z - A_zB_y)\mathbf{e}_x + (A_zB_x - A_xB_z)\mathbf{e}_y + (A_xB_y - A_yB_x)\mathbf{e}_z, \tag{1.5}$$

where $\mathbf{e}_n$ is the unit vector which is perpendicular to the plane formed by $\mathbf{A}$ and $\mathbf{B}$ and lies in the direction indicated by the right-hand rule if $\mathbf{A}$ is rotated into $\mathbf{B}$. A unit vector is a vector with unit length. We recall that $\mathbf{B}\times\mathbf{A} = -\mathbf{A}\times\mathbf{B}$.

1.2 The Gradient Operator

The vector operator *del* is defined in Cartesian coordinates as

$$\nabla = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}. \tag{1.6}$$

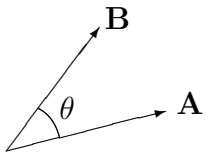


Figure 1: Two Vectors.

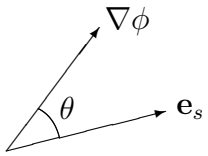


Figure 2: The Directional Derivative.

The *gradient* of a scalar function $\phi(x, y, z)$ is defined as $\nabla\phi$, so that, in Cartesian coordinates,

$$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{e}_x + \frac{\partial\phi}{\partial y}\mathbf{e}_y + \frac{\partial\phi}{\partial z}\mathbf{e}_z. \tag{1.7}$$

This vector has as its components the rates of change of ϕ with respect to distance in the x, y, and z directions, respectively.

The *directional derivative* measures the rate of change of a scalar function, say $\phi(x, y, z)$, with respect to distance in any arbitrary direction s (Fig. 2) and is given by

$$\frac{\partial\phi}{\partial s} = \mathbf{e}_s \cdot \nabla\phi = |\mathbf{e}_s||\nabla\phi| \cos\theta = |\nabla\phi| \cos\theta, \tag{1.8}$$

where $\mathbf{e}_s$ is the unit vector in the s direction, and θ is the angle between the two vectors $\nabla\phi$ and $\mathbf{e}_s$. Thus, the maximum rate of change of ϕ is in the direction of $\nabla\phi$.

Given a vector function (field) $\mathbf{A}(x, y, z)$ (e.g., velocity), the *divergence* of $\mathbf{A}$ is defined as $\nabla \cdot \mathbf{A}$, so that, in Cartesian coordinates,

$$\begin{aligned}\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} &= \left(\mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}\right) \cdot (A_x\mathbf{e}_x + A_y\mathbf{e}_y + A_z\mathbf{e}_z) \\ &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.\end{aligned} \tag{1.9}$$

The *curl* of the vector field $\mathbf{A}$ is defined as $\nabla \times \mathbf{A}$, so that, in Cartesian coordinates,

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \tag{1.10}$$

In fluid dynamics, for example, the curl of the velocity field is defined as the vorticity.

There are several useful vector identities involving the gradient operator:

$$\nabla \cdot (\phi\mathbf{f}) = \nabla\phi \cdot \mathbf{f} + \phi\,\nabla \cdot \mathbf{f}, \tag{1.11}$$

$$\nabla \times (\phi\mathbf{f}) = \nabla\phi \times \mathbf{f} + \phi\,\nabla \times \mathbf{f}, \tag{1.12}$$

$$\nabla \times \nabla\phi = \mathbf{0}, \tag{1.13}$$

$$\nabla \cdot (\nabla \times \mathbf{u}) = 0, \tag{1.14}$$

$$\nabla \cdot (\nabla\phi_1 \times \nabla\phi_2) = 0. \tag{1.15}$$

The first two of these identities can be derived using the product rule for derivatives. Eqs. 1.13 and 1.14 state that the curl of the gradient and the divergence of the curl both vanish.

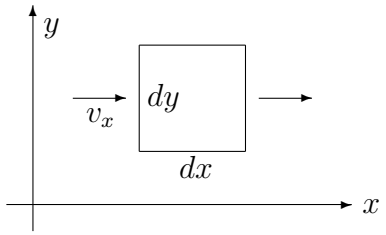


Figure 3: Fluid Flow Through Small Parallelepiped.

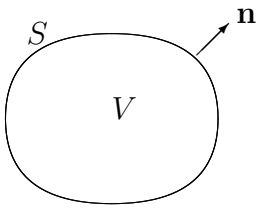


Figure 4: A Closed Volume Bounded by a Surface S.

1.3 The Divergence Theorem

Consider the steady (time-independent) motion of a fluid of density $\rho(x, y, z)$ and velocity

$$\mathbf{v} = v_x(x, y, z)\mathbf{e}_x + v_y(x, y, z)\mathbf{e}_y + v_z(x, y, z)\mathbf{e}_z. \tag{1.16}$$

In the small parallelepiped of dimensions dx, dy, dz (Fig. 3)[13], the mass entering the face $dy\,dz$ on the left per unit time is $\rho v_x\,dy\,dz$. The mass exiting per unit time on the right is given by the two-term Taylor series expansion as

$$\left[\rho v_x + \frac{\partial(\rho v_x)}{\partial x}\,dx\right] dy\,dz,$$

so that the loss of mass per unit time in the x-direction is

$$\frac{\partial}{\partial x}(\rho v_x)\,dx\,dy\,dz.$$

If we also take into consideration the other two directions, the total loss of mass per unit time for the small parallelepiped is

$$\left[\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z)\right] dx\,dy\,dz = \nabla\cdot(\rho\mathbf{v})\,dV.$$

If we now consider a volume V bounded by a closed surface S (Fig. 4), the total loss of mass per unit time is

$$\int_V \nabla\cdot(\rho\mathbf{v})\,dV.$$

However, the loss of fluid in V must be due to the flow of fluid though the boundary S. The outward flow of mass per unit time through a differential element of surface area dS is $\rho\mathbf{v}\cdot\mathbf{n}\,dS$, so that the total loss of mass per unit time through the boundary is

$$\oint_S \rho\mathbf{v}\cdot\mathbf{n}\,dS,$$

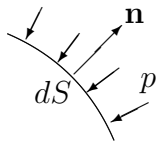


Figure 5: Pressure on Differential Element of Surface.

where $\mathbf{n}$ is the unit outward normal at a point on S, $\mathbf{v} \cdot \mathbf{n}$ is the normal component of $\mathbf{v}$, and the circle in the integral sign indicates that the surface integral is for a closed surface. Equating the last two expressions, we have

$$\int_V \nabla \cdot (\rho \mathbf{v})\, dV = \oint_S \rho \mathbf{v} \cdot \mathbf{n}\, dS, \tag{1.17}$$

or, in general, for any vector field $\mathbf{f}(x, y, z)$ defined in a volume V bounded by a closed surface S,

$$\int_V \nabla \cdot \mathbf{f}\, dV = \oint_S \mathbf{f} \cdot \mathbf{n}\, dS. \tag{1.18}$$

This is the *divergence theorem*, one of the fundamental integral theorems of mathematical physics. The divergence theorem is also known as the Green-Gauss theorem and the Ostrogradski theorem.

Related theorems are the *gradient theorem*, which states that

$$\int_V \nabla f\, dV = \oint_S f \mathbf{n}\, dS, \tag{1.19}$$

for f a scalar function, and the *curl theorem*:

$$\int_V \nabla \times \mathbf{f}\, dV = \oint_S \mathbf{n} \times \mathbf{f}\, dS. \tag{1.20}$$

To aid in the interpretation of the divergence of a vector field, consider a small volume ΔV surrounding a point P. From the divergence theorem,

$$(\nabla \cdot \mathbf{f})_P \approx \frac{1}{\Delta V} \oint_S \mathbf{f} \cdot \mathbf{n}\, dS, \tag{1.21}$$

which implies that the divergence of $\mathbf{f}$ can be interpreted as the net outward flow (flux) of $\mathbf{f}$ at P per unit volume. (If $\mathbf{f}$ is the velocity field $\mathbf{v}$, the right-hand side integrand is the normal component v_n of velocity.) Similarly, we can see the plausibility of the divergence theorem by observing that the divergence multiplied by the volume element for each elemental cell is the net surface integral out of that cell. When summed by integration, all internal contributions cancel, since the flow out of one cell goes into another, and only the external surface contributions remain.

To illustrate the use of the integral theorems, consider a solid body of arbitrary shape to which the non-uniform pressure $p(x, y, z)$ is applied to the surface. The force on a differential element of surface dS (Fig. 5) is $p\, dS$. Since this force acts in the $-\mathbf{n}$ direction, the resultant

force $\mathbf{F}$ acting on the body is then

$$\mathbf{F} = -\oint_S p\mathbf{n}\, dS = -\int_V \nabla p\, dV, \tag{1.22}$$

where the second equation follows from the gradient theorem. This relation is quite general, since it applies to arbitrary pressure distributions. For the special case of uniform pressure (p = constant), $\mathbf{F} = \mathbf{0}$; i.e., the resultant force of a uniform pressure load on an arbitrary body is zero.

A variation of the divergence theorem results if we define $\mathbf{f} = u\nabla w$ for two scalar functions u and w. Substituting this function into the divergence theorem, Eq. 1.18, we obtain Green's Theorem of the First Kind,

$$\int_V (\nabla u \cdot \nabla w + u\nabla^2 w)\, dV = \oint_S u\frac{\partial w}{\partial n}\, dS, \tag{1.23}$$

where ∇^2 is referred to as the Laplacian operator and given by

$$\nabla^2 w = \nabla \cdot \nabla w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \tag{1.24}$$

in Cartesian coordinates, and

$$\frac{\partial w}{\partial n} = \nabla w \cdot \mathbf{n} \tag{1.25}$$

is the normal derivative of w at a point on the surface S. Green's Theorem of the Second Kind is obtained by interchanging u and w in Eq. 1.23 and subtracting the result from Eq. 1.23:

$$\int_V (u\nabla^2 w - w\nabla^2 u)\, dV = \oint_S \left(u\frac{\partial w}{\partial n} - w\frac{\partial u}{\partial n}\right) dS. \tag{1.26}$$

1.4 The Dirac Delta Function

A useful function in a variety of situations in applied mathematics and mathematical physics is the Dirac delta function $\delta(t)$, which is defined as the function with the properties

$$\delta(t - t_0) = 0 \text{ for } t \neq t_0, \tag{1.27}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\, dt = f(t_0). \tag{1.28}$$

In particular, when $f(t) = 1$,

$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1. \tag{1.29}$$

Thus, δ can be thought of as a spike of infinite height, infinitesimal width, and unit area under the curve, as shown in Fig. 6. To be able to differentiate the delta function, it is more convenient to think of the function as a smooth function, as indicated in Fig. 6b. Strictly speaking, δ is not a function in the usual sense, but it has been legalized as a *distribution*

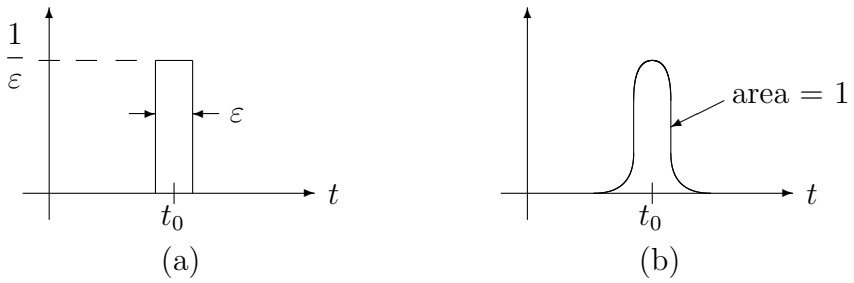


Figure 6: The Dirac Delta Function.

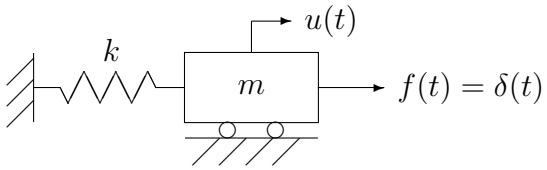


Figure 7: Mass-Spring System.

or *generalized function*, which means that δ is known by its effect on smooth functions $f(x)$ [9].

To illustrate the use of the Dirac delta function, consider the ordinary differential equation for a one-dimensional mass-spring system (Fig. 7) initially at rest and subjected to an impulsive load:

$$m\ddot{u} + ku = \delta(t), \quad u(0) = \dot{u}(0) = 0, \tag{1.30}$$

where u is displacement, t is time, m is mass, k is stiffness, and dots denote derivatives with respect to the time t. The time-dependent loading $\delta(t)$ is referred to as the unit impulse function in mechanics. Since the right-hand side of Eq. 1.30a is nonzero only for $t = 0$, we integrate that equation from $t = 0-$ to $t = 0+$ to obtain

$$\int_{0-}^{0+} m\ddot{u}\, dt + \int_{0-}^{0+} ku\, dt = \int_{0-}^{0+} \delta(t)\, dt = 1. \tag{1.31}$$

Since integration is a smoothing process (as illustrated in Fig. 8 for the delta function), the right-hand side singularity in Eq. 1.30a must be exhibited by the highest order derivative appearing on the left-hand side. If $\ddot{u}$ resembles a Dirac delta at $t = 0$, $\dot{u}$ would have a simple

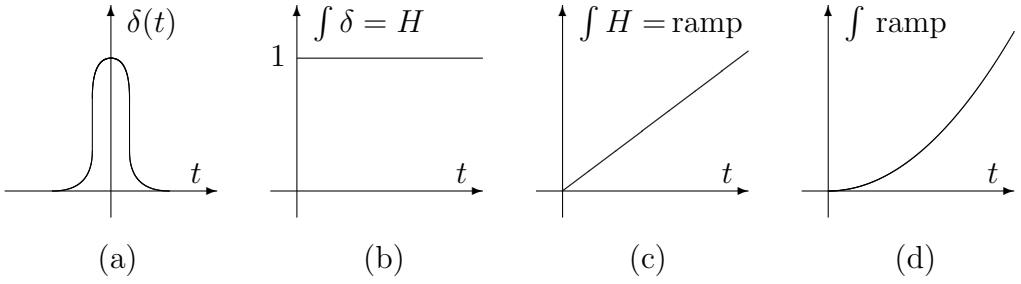


Figure 8: Successive Integrals of the Delta Function.

jump discontinuity at $t = 0$, and u (and its integral) would be continuous at $t = 0$. Thus, from Eq. 1.31,

$$m[\dot{u}(0+) - \dot{u}(0-)] + 0 = 1. \tag{1.32}$$

Since the initial velocity vanishes, $\dot{u}(0+) = 1/m$. Thus we have shown that specifying an impulsive load to a mechanical system at rest is equivalent to specifying a nonzero initial velocity to the system with no load:

$$m\ddot{u} + ku = 0, \quad u(0) = 0, \quad \dot{u}(0) = 1/m. \tag{1.33}$$

Eqs. 1.30 and 1.33 are equivalent. The response to the unit impulse function $\delta(t)$ is called the unit impulse response and denoted $h(t)$.

The usefulness of the impulse response is that it completely characterizes the system and can be used to determine the response to arbitrary forcing functions. For example, the solution $u(t)$ of the system

$$m\ddot{u} + ku = f(t), \quad u(0) = \dot{u}(0) = 0, \tag{1.34}$$

where $f(t)$ is arbitrary, can be seen by direct substitution to be the convolution integral

$$u(t) = \int_0^t f(\tau)h(t-\tau)\,d\tau, \tag{1.35}$$

where $h(t)$ satisfies Eq. 1.30. The differentiation of the integral, Eq. 1.35, is effected with the aid of Leibnitz's rule:

$$\frac{d}{dx}\int_{A(x)}^{B(x)} f(x,t)\,dt = \int_A^B \frac{\partial f(x,t)}{\partial x}\,dt + f(x,B)\frac{dB}{dx} - f(x,A)\frac{dA}{dx}. \tag{1.36}$$

Although we used the delta function here for an impulsive function, delta functions are also used for spacially local (concentrated) forcing functions. With partial differential equations, the response to a localized input (δ) is called a Green's function.

2 Derivation and Classification of PDEs

A *partial differential equation* is an equation that involves an unknown function and some of its partial derivatives with respect to two or more independent variables. An nth order equation has its highest order derivative of order n. A partial differential equation is *linear* if it is an equation of the first degree in the dependent variable and its derivatives. A partial differential equation is *homogeneous* if every term contains the dependent variable or one of its partial derivatives. The trivial (zero) function is always a solution of a homogeneous equation.

Most of the interest in this volume is in linear second-order equations, the most general of which in two independent variables is

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{2.1}$$

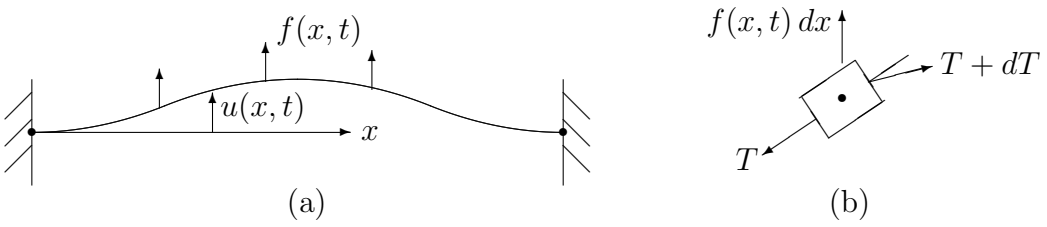


Figure 9: The Vibrating String.

where the coefficients are functions of the independent variables x and y [i.e., $A = A(x, y)$, $B = B(x, y)$, etc.], and we use the subscript notation to denote partial derivatives:

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \; u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \; \text{etc.} \tag{2.2}$$

Eq. 2.1 is homogeneous if $G(x, y) = 0$. We now proceed to derive the partial differential equations for several problems in physics.

2.1 The Vibrating String

Consider a string in tension between two fixed end-points and acted upon by transverse forces, as shown in Fig. 9. Let $u(x, t)$ denote the transverse displacement, where x is the distance from the left end, and t is time. The force distribution (of dimension force/length) is $f(x, t)$. We assume that all motion is vertical, and the displacement u and slope $\theta = \partial u/\partial x$ are both small. (For small θ, $\sin\theta \approx \tan\theta$.) The differential equation of motion is derived by applying Newton's second law ($F = ma$) to a small differential segment of string (Fig. 9b). If the string tension is T at some location x, the tension a short distance dx away is obtained using the two-term Taylor's series approximation

$$T + dT \approx T + \frac{\partial T}{\partial x} dx. \tag{2.3}$$

Similarly, given the slope $\partial u/\partial x$ at x, the slope at $x + dx$ is approximately

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) dx = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx. \tag{2.4}$$

Using Newton's Second Law, we now equate the net applied force in the vertical direction to ma (mass $\times$ acceleration):

$$\left(T + \frac{\partial T}{\partial x} dx\right)\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx\right) - T\frac{\partial u}{\partial x} + f(x, t)\, dx = (\rho A\, dx)\frac{\partial^2 u}{\partial t^2}, \tag{2.5}$$

where ρ is the density of the string material, and A is the cross-sectional area. After expanding this equation and eliminating small terms, we obtain the linearized partial differential equation for the vibrating string

$$T\frac{\partial^2 u}{\partial x^2} + f(x, t) = \rho A \frac{\partial^2 u}{\partial t^2} \tag{2.6}$$

or

$$u_{xx} + \frac{f(x,t)}{T} = \frac{1}{c^2} u_{tt}, \tag{2.7}$$

where

$$c = \sqrt{\frac{T}{\rho A}}. \tag{2.8}$$

Note that the constant c has the units of velocity. For zero force ($f = 0$), Eq. 2.7 reduces to

$$u_{xx} = u_{tt}/c^2, \tag{2.9}$$

which is the one-dimensional wave equation. Thus, the transverse displacement for the unforced, infinitesimal vibrations of a stretched string satisfies the wave equation.

In two space dimensions, the wave equation becomes

$$u_{xx} + u_{yy} = u_{tt}/c^2, \tag{2.10}$$

which is applicable to the transverse vibrations of a stretched membrane (e.g., a drum head). In this case, the velocity c is given by

$$c = \sqrt{\frac{T}{m}}, \tag{2.11}$$

where T is the tension per unit length, m is the mass per unit area (i.e., $m = \rho h$), and h is the membrane thickness. The membrane is the two-dimensional analog of the string, since, in both cases, there is no flexural rigidity.

In three dimensions, the wave equation becomes

$$u_{xx} + u_{yy} + u_{zz} = u_{tt}/c^2, \tag{2.12}$$

which is the applicable equation for acoustics, where the variable u represents, for example, the pressure p or the velocity potential ϕ, and c is the speed of sound given by

$$c = \sqrt{\frac{B}{\rho}}, \tag{2.13}$$

where $B = \rho c^2$ is the fluid bulk modulus, and ρ is the density. Note that, in three space dimensions, the dependent variable $u(x, y, z, t)$ depends on four independent variables.

In any number of dimensions, the wave equation can therefore be written

$$\nabla^2 u = \frac{1}{c^2} u_{tt}, \tag{2.14}$$

where the left-hand side $\nabla^2 u$ (the Laplacian of u) is $u_{xx} + u_{yy} + u_{zz}$ in 3-D, $u_{xx} + u_{yy}$ in 2-D, and u_{xx} in 1-D.

We mention without proof here that an example of a well-posed boundary-initial value problem (BIVP) is

$$u_{xx} = \frac{1}{c^2} u_{tt}, \ 0 < x < L \tag{2.15}$$

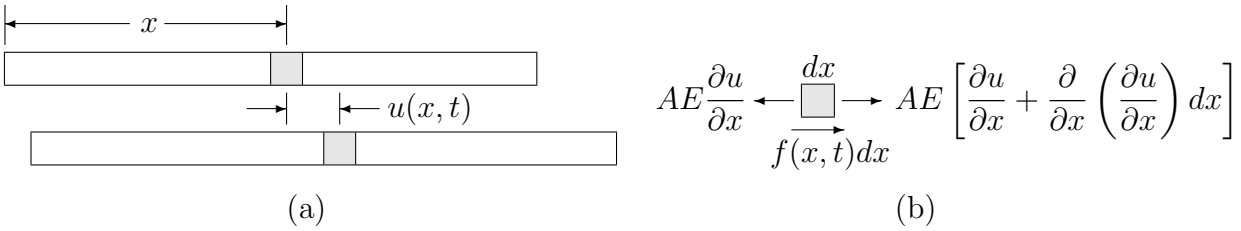


Figure 10: Longitudinal Vibrations of Bar.

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0 \tag{2.16}$$

and the initial conditions

$$u(x,0) = g(x),\ u_t(x,0) = 0. \tag{2.17}$$

The above system describes a string released from rest from an arbitrary initial displaced shape $g(x)$. *Well-posed* means that the solution exists, is unique, and depends continuously on the data (the various parameters of the problem, including geometry, material properties, boundary conditions, and initial conditions).

For time-harmonic motion, $u = u_0 \cos(\omega t)$, and the wave equation, Eq. 2.14, simplifies to the Helmholtz equation,

$$\nabla^2 u_0 + k^2 u_0 = 0, \tag{2.18}$$

where u_0 is the amplitude of the sine wave, and $k = \omega/c$ is called the wave number. The Helmholtz equation is sometimes referred to as the *reduced wave equation.*

2.2 Longitudinal Vibrations of Bars

Consider a uniform bar subjected to a time-dependent axial force $f(x,t)$ (of dimension force/length). This bar undergoes longitudinal vibrations, as indicated in Fig. 10. The longitudinal displacement at a typical point x is denoted $u(x,t)$. Strain at a point in the bar is $\partial u/\partial x$, stress is $E\partial u/\partial x$, and the force at a given cross-section is $AE\partial u/\partial x$, where A is the cross-sectional area of the bar, and E is the Young's modulus of the bar material. The dynamic force balance of an infinitesimal segment of length dx (Fig. 10b) is, from Newton's Second Law ($F = ma$),

$$f(x,t)\,dx + AE\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)dx = (\rho A\,dx)\frac{\partial^2 u}{\partial t^2}, \tag{2.19}$$

or

$$u_{xx} + \frac{f(x,t)}{AE} = \frac{1}{c^2}u_{tt}, \tag{2.20}$$

where

$$c = \sqrt{\frac{E}{\rho}}. \tag{2.21}$$

The equation of motion for the longitudinal vibrations of a bar is thus identical to that of the transverse vibrations of a stretched string, Eq. 2.7. The only difference is the definition of the velocity c. For zero force ($f = 0$), Eq. 2.20 reduces to

$$u_{xx} = u_{tt}/c^2, \tag{2.22}$$

which is the one-dimensional wave equation. Thus, the longitudinal displacement for the unforced, infinitesimal, longitudinal vibrations of a uniform bar satisfies the wave equation.

This derivation was shorter than that for the string because we made the implicit assumption that the area A was constant. However, the Poisson's ratio effect implies that, if the stress varies with position, the area A must also change. For small motions, the variation of A with position is analogous to that of the tension T in the string problem. Both effects are second-order effects which are ignored for small motions.

2.3 Heat Conduction in Solids

The basis for heat conduction analysis is Fourier's law of heat conduction, which states that the rate $\hat{q}$ of heat flow by conduction in a given direction (say, the x direction) is proportional to the temperature gradient dT/dx and the cross-sectional area A through which the heat flows:

$$\hat{q}_x = -kA\frac{dT}{dx}, \tag{2.23}$$

where T is temperature. The proportionality constant k is the *thermal conductivity*. The minus sign in this equation indicates that heat (energy) flows from hot to cold. Since the units for heat are joules in the International System of Units (SI), the units for $\hat{q}_x$ are J/s. The units for conductivity k, a derived quantity, are therefore J/(s-m-°C).

To derive the differential equation for heat conduction, we first define the rate of heat flow per unit area $q_x = \hat{q}_x/A$. In general, combining similar expressions for all three Cartesian directions, we define the heat flux vector $\mathbf{q}$ as

$$\mathbf{q} = q_x\mathbf{e}_x + q_y\mathbf{e}_y + q_z\mathbf{e}_z = -k\frac{\partial T}{\partial x}\mathbf{e}_x - k\frac{\partial T}{\partial y}\mathbf{e}_y - k\frac{\partial T}{\partial z}\mathbf{e}_z, \tag{2.24}$$

or

$$\mathbf{q} = -k\nabla T. \tag{2.25}$$

Since we have used the same conductivity k for all three directions, material isotropy has been assumed. The heat flux vector $\mathbf{q}$ has the units of energy per unit time per unit area.

Now consider an arbitrary body of volume V, closed surface S, and unit outward normal $\mathbf{n}$ (Fig. 11). The balance of energy for this body is that the heat entering the body through the surface per unit time plus the heat produced by sources in the body per unit time equals the time rate of change of the heat content of the body. That is, during the time interval Δt,

$$\left(\oint_S -\mathbf{q}\cdot\mathbf{n}\,dS + \int_V Q\,dV\right)\Delta t = \int_V \rho c\Delta T\,dV, \tag{2.26}$$

where $Q(x, y, z)$ is the internal heat generation per unit volume per unit time, ρ is the material density, c is the specific heat (the heat required per unit mass to raise the temperature by

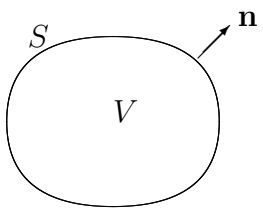


Figure 11: Arbitrary Body of Volume V and Surface S.

one degree), and t is time. In the limit $\Delta t \to 0$,

$$\oint_S -\mathbf{q} \cdot \mathbf{n}\, dS + \int_V Q\, dV = \int_V \rho c \frac{\partial T}{\partial t}\, dV. \tag{2.27}$$

We now substitute Fourier's law of heat conduction, Eq. 2.25, into Eq. 2.27, and apply the divergence theorem, Eq. 1.18, to obtain

$$\int_V \left[\nabla \cdot (k \nabla T) + Q - \rho c \frac{\partial T}{\partial t} \right] dV = 0. \tag{2.28}$$

Since this equation holds for arbitrary volumes, then, at points of continuity of the integrand, the integrand must vanish, thus yielding the heat conduction equation

$$\nabla \cdot (k \nabla T) + Q = \rho c \frac{\partial T}{\partial t} \quad \text{in } V, \tag{2.29}$$

or, in Cartesian coordinates,

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + Q = \rho c \frac{\partial T}{\partial t}. \tag{2.30}$$

This equation assumes that ρc is independent of temperature and that the material is isotropic. This use of the divergence theorem is also fundamental in the derivation of differential equations in continuum mechanics, where surface integrals involving surface tractions are converted to volume integrals.

We have interest in several special cases of Eq. 2.29. For a homogeneous material, k is independent of position, and Eq. 2.29 simplifies to

$$k \nabla^2 T + Q = \rho c \frac{\partial T}{\partial t}, \tag{2.31}$$

or

$$\nabla^2 T = \frac{1}{K} \frac{\partial T}{\partial t} - \frac{Q}{k}, \tag{2.32}$$

where $K = k/(\rho c)$ is the *thermal diffusivity*. If the system is also steady (independent of time),

$$\nabla^2 T = -Q/k, \tag{2.33}$$

which is Poisson's equation. The steady-state temperature in a system with a homogeneous material and no sources ($Q = 0$) satisfies Laplace's equation

$$\nabla^2 T = 0. \tag{2.34}$$

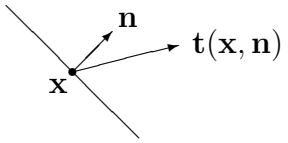


Figure 12: Stress Vector.

This equation is also known as the potential equation, and solutions are called *harmonic* functions. One additional special case occurs with one-dimensional heat conduction in a homogeneous rod, for which the temperature satisfies

$$\frac{\partial^2 T}{\partial x^2} = \frac{\rho c}{k}\frac{\partial T}{\partial t} - \frac{Q}{k}. \tag{2.35}$$

We have now encountered five classical equations of mathematical physics: Laplace's equation (Eq. 2.34), Poisson's equation (Eq. 2.33), the wave equation (Eq. 2.14), the heat equation (Eq. 2.29), and the Helmholtz equation (Eq. 2.18).

2.4 Elastodynamics

Consider a general three-dimensional elastic body (Fig. 11) acted upon by forces. There are two kinds of forces of interest:

1. body forces, which act at each point throughout the volume of the body (e.g., gravity), and

2. surface forces (tractions), which act only on surface points (e.g., hydrostatic pressure on a submerged body).

If $\mathbf{f}$ denotes the body force per unit mass, the net force on the body due to $\mathbf{f}$ is

$$\int_V \rho \mathbf{f}\, dV.$$

Let $\mathbf{t}(\mathbf{x}, \mathbf{n})$ denote the stress vector, the internal force per unit area on an internal surface (at a point $\mathbf{x}$) with normal $\mathbf{n}$ (Fig. 12). This vector depends on the orientation of the surface and has three components. We define $t_j(\mathbf{x}, \mathbf{e}_i) = \sigma_{ij}$ as the jth component of the stress vector acting on a surface with normal in the ith direction. In three dimensions, i and j take the values $1, 2, 3$. For a general normal $\mathbf{n}$, the ith component of $\mathbf{t}$ is related to the stress components σ_{ij} by

$$t_i = \sum_{j=1}^{3} \sigma_{ji} n_j. \tag{2.36}$$

We could alternatively use index notation with the Einstein summation convention, where twice-repeated indices are summed over the range, and the summation symbol is discarded, and simply write

$$t_i = \sigma_{ji} n_j. \tag{2.37}$$

Other common operations have compact representation using index notation. For example, the scalar (dot) product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{3} a_i b_i = a_i b_i, \tag{2.38}$$

and the divergence of a vector field $\mathbf{A}$ can be written in Cartesian coordinates as

$$\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = A_{i,i}, \tag{2.39}$$

where partial derivatives are denoted with commas. For example,

$$\frac{\partial \phi}{\partial x_i} = \phi_{,i}. \tag{2.40}$$

Consequently, the divergence theorem, Eq. 1.18, can be written in index notation as

$$\int_V f_{i,i} \, dV = \oint_S f_i n_i \, dS \tag{2.41}$$

for any vector field $\mathbf{f}$.

To derive the equations of dynamic equilibrium for a body, we use Newton's second law, which states that the net force on a body (the sum of surface and body forces) equals the product of mass and acceleration:

$$\oint_S \mathbf{t} \, dS + \int_V \rho \mathbf{f} \, dV = \int_V \ddot{\mathbf{u}} \rho \, dV \tag{2.42}$$

or, in index notation,

$$\oint_S t_i \, dS + \int_V \rho f_i \, dV = \int_V \rho \ddot{u}_i \, dV, \tag{2.43}$$

where dots denote time derivatives. The first term of this equation is given by

$$\oint_S t_i \, dS = \oint_S \sigma_{ji} n_j \, dS = \int_V \sigma_{ji,j} \, dV, \tag{2.44}$$

where the second equality follows from the divergence theorem, Eq. 2.41. Thus, from Eq. 2.43,

$$\int_V (\sigma_{ji,j} + \rho f_i - \rho \ddot{u}_i) \, dV = 0. \tag{2.45}$$

Since this equation must hold for all V, the integrand must vanish at points of continuity of the integrand:

$$\sigma_{ji,j} + \rho f_i = \rho \ddot{u}_i, \tag{2.46}$$

which is referred to as the momentum equation, since it arose from the balance of linear momenta.

It is also shown in the theory of elasticity [21] that, from the balance of moments,

$$\sigma_{ij} = \sigma_{ji} \tag{2.47}$$

(i.e., the stress tensor σ_{ij} is symmetric). Thus, Eqs. 2.46 and 2.37 can be written

$$\sigma_{ij,j} + \rho f_i = \rho \ddot{u}_i, \qquad t_i = \sigma_{ij} n_j. \tag{2.48}$$

The momentum equation, which is expressed in terms of the internal forces (stresses) on a body, says nothing about the resulting deformation. The linearized strain (a dimensionless quantity representing the change in length per unit length) can be written in terms of the displacement gradients (in index notation) as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.49}$$

where u_1, u_2, and u_3 denote the three Cartesian components of displacement (u, v, w), and x_1, x_2, and x_3 denote the three Cartesian coordinates (x, y, z). Note that the strain tensor ε_{ij} is symmetric, so that there are only six unique components of strain. Thus, for a general body undergoing the deformation

$$\begin{cases} u = u(x, y, z), \\ v = v(x, y, z), \\ w = w(x, y, z), \end{cases} \tag{2.50}$$

the six strain components are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \tag{2.51}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}, \tag{2.52}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}, \tag{2.53}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\gamma_{xy}, \tag{2.54}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\gamma_{yz}, \tag{2.55}$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\gamma_{xz}, \tag{2.56}$$

where γ_{ij} denotes the engineering shear strains. (The shear strains ε_{ij} are used in the mathematical theory of elasticity rather than γ_{ij}, because of the conveniences which result from the fact that ε_{ij} is a tensor of rank 2.)

The stress components σ_{ij} are related to the strain components ε_{ij} by generalized Hooke's Law, which states that each stress component is a linear combination of all the strain components. For general anisotropy, the six components of stress are thus related to the six components of strain by at most 21 experimentally-measured elastic constants. For an isotropic material (for which all directions are equivalent), the number of independent elastic constants reduces to two, and Hooke's Law can be written in the form

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} = \lambda u_{k,k} \delta_{ij} + \mu(u_{i,j} + u_{j,i}), \tag{2.57}$$

where λ and μ are the Lamé constants of elasticity, and δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{2.58}$$

It can be shown [21] that the Lamé constants are related to the engineering constants E (Young's modulus) and ν (Poisson's ratio) by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{2.59}$$

or, inversely,

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}. \tag{2.60}$$

The Lamé constant μ is the shear modulus G.

The substitution of Hooke's Law, Eq. 2.57, into the first term of the momentum equation, Eq. 2.48a, yields

$$\sigma_{ij,j} = \lambda u_{k,kj}\delta_{ij} + \mu(u_{i,jj} + u_{j,ij}) = \lambda u_{k,ki} + \mu(u_{i,jj} + u_{j,ij}) = \mu u_{i,jj} + (\lambda+\mu)u_{j,ji}. \tag{2.61}$$

Thus, the equations of motion are

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ji} + \rho f_i = \rho\ddot{u}_i \quad (i = 1, 2, 3). \tag{2.62}$$

These three equations are referred to as the Navier equations of elasticity. For example, the x-component of this equation is

$$(\lambda+2\mu)u_{,xx} + \mu(u_{,yy} + u_{,zz}) + (\lambda+\mu)(v_{,xy} + w_{,xz}) + \rho f_x = \rho\ddot{u}, \tag{2.63}$$

where u, v, and w are the Cartesian components of displacement, and commas denote partial differentiation. Similar equations in the y and z directions thus yield a system of three coupled partial differential equations in the three unknowns u, v, and w.

2.5 Flexural Vibrations of Beams

To derive the differential equation for the flexural vibration of beams, consider the forces and moments acting on the differential element shown in Fig. 13. The application of Newton's second law ($F = ma$) in the y-direction yields [5]

$$V - \left(V + \frac{\partial V}{\partial x}\,dx\right) + p(x,t)\,dx = m\frac{\partial^2 u}{\partial t^2}\,dx \tag{2.64}$$

or

$$\frac{\partial V}{\partial x} + m\frac{\partial^2 u}{\partial t^2} = p(x,t), \tag{2.65}$$

where V is the shear force, M is the bending moment, $p(x,t)$ is the transverse loading (force/length), $m(x)$ is the mass per unit length, and $u(x,t)$ is the transverse displacement.

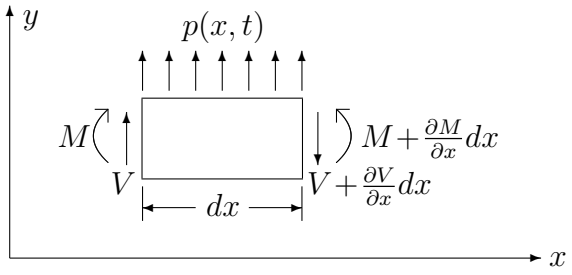


Figure 13: Forces on Beam Element.

By summing moments about a point on the right face of the element, we obtain

$$M + V\,dx + \frac{1}{2}\,p(x,t)(dx)^2 = M + \frac{\partial M}{\partial x}\,dx \tag{2.66}$$

so that, in the limit (small dx),

$$V = \frac{\partial M}{\partial x}, \tag{2.67}$$

which is the usual static relationship. The deformations due to shear and the effects of rotatory inertia have both been ignored. With the basic moment-curvature relationship of elementary beam theory,

$$M = EI\frac{\partial^2 u}{\partial x^2}, \tag{2.68}$$

we obtain

$$\frac{\partial V}{\partial x} = \frac{\partial^2 M}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right), \tag{2.69}$$

where EI is the flexural rigidity of the beam, E is the Young's modulus of the beam material, and I is the second moment of inertia of the beam cross-section. Thus, the partial differential equation of motion for flexural vibrations is

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2}\right) + m\frac{\partial^2 u}{\partial t^2} = p(x,t), \tag{2.70}$$

where both EI and m may vary arbitrarily with x. For a uniform beam,

$$EI\frac{\partial^4 u}{\partial x^4} + m\frac{\partial^2 u}{\partial t^2} = p(x,t). \tag{2.71}$$

Since this equation is second-order in time, two initial conditions (e.g., initial displacement and velocity) would need to be specified for uniqueness of the solution. In addition, since the equation is fourth-order in x, four boundary conditions would need to be specified. For example, for a cantilever beam of length L fixed at $x = 0$, the four boundary conditions would be

$$u(0,t) = \frac{\partial u(0,t)}{\partial x} = 0, \quad M(L,t) = EI\frac{\partial^2 u(L,t)}{\partial x^2} = 0, \quad V(L,t) = EI\frac{\partial^3 u(L,t)}{\partial x^3} = 0. \tag{2.72}$$

For free (unforced), time-harmonic motion, $p(x,t) = 0$, and

$$u(x,t) = v(x)\cos\omega t, \tag{2.73}$$

so that Eq. 2.71 yields the eigenvalue problem

$$EI\frac{d^4v}{dx^4} = m\omega^2 v \tag{2.74}$$

or

$$\frac{d^4v}{dx^4} = a^4 v, \tag{2.75}$$

where

$$a = \sqrt[4]{\frac{m\omega^2}{EI}}. \tag{2.76}$$

The general solution of Eq. 2.75, an ordinary differential equation, is

$$v(x) = c_1 \sin ax + c_2 \cos ax + c_3 \sinh ax + c_4 \cosh ax, \tag{2.77}$$

where the frequency parameter a (of dimension 1/length) and the four constants c_1, c_2, c_3, c_4 can be determined from the boundary conditions.

2.6 Boundary Conditions

There are several types of boundary conditions commonly encountered with partial differential equations:

1. Dirichlet boundary conditions, which specify the value of the function on the boundary, e.g.,
$$T = f(x, y, z, t) \text{ on } S, \tag{2.78}$$

2. Neumann boundary conditions, which specify the normal derivative of the function on the boundary, e.g.,
$$\frac{\partial T}{\partial n} = g(x, y, z, t) \text{ on } S, \tag{2.79}$$

3. Robin boundary conditions, which specify a linear combination of the unknown and its normal derivative on the boundary, e.g.,
$$a\frac{\partial T}{\partial n} + bT = f(x, y, z, t) \text{ on } S. \tag{2.80}$$

Different parts of the boundary can have different types of boundary conditions. In a heat transfer problem, for example, the Dirichlet boundary condition specifies the temperature on the boundary, the Neumann boundary condition specifies the heat flux on the boundary, and the Robin boundary condition could occur on a boundary for which heat transfer occurs due to thermal convection (i.e., heat flux is proportional to temperature). Robin boundary conditions are sometimes referred to as impedance boundary conditions, since, in mechanics and acoustics, the ratio of the gradient of an unknown to that unknown is referred to as impedance. (Acoustic impedance is defined as pressure divided by velocity, and mechanical impedance is typically defined as force divided by velocity.)

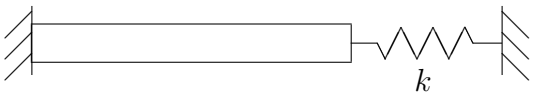


Figure 14: Bar With Spring Boundary Condition.

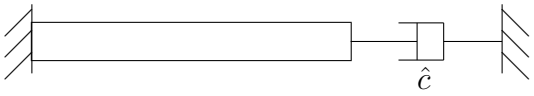


Figure 15: Bar With Dashpot Boundary Condition.

A mechanical example of a Robin boundary condition is provided by a rod with a spring attached to the end, as shown in Fig. 14. The internal longitudinal force in the bar is given by $AE(\partial u/\partial x)$, where A is the cross-sectional area of the bar, E is the Young's modulus of the bar material, u is longitudinal displacement, and $\partial u/\partial x$ is strain. At the end, this force is balanced by the spring force ku, where k is the spring constant. Thus, at the end of the bar, the boundary condition provided by the spring is the impedance (Robin) condition

$$AE\frac{\partial u}{\partial x} + ku = 0. \tag{2.81}$$

Notice that the two terms in this equation have the same sign, so that, when the end displacement u is positive, the end of the bar is in compression, and the strain $\partial u/\partial x$ is negative. By analogy with this mechanical case, we can also conclude, in general, that the the two factors a and b in Eq. 2.80 must have the same sign if the boundary condition is to make sense physically.

Similarly, if the end spring is replaced by a dashpot (Fig. 15), whose force is proportional to velocity, the boundary condition on the end is

$$AE\frac{\partial u}{\partial x} + \hat{c}\frac{\partial u}{\partial t} = 0. \tag{2.82}$$

If both a spring and a dashpot were attached at the end in parallel, the boundary condition would be

$$AE\frac{\partial u}{\partial x} + \hat{c}\frac{\partial u}{\partial t} + ku = 0. \tag{2.83}$$

If a lumped mass m were also attached to the end point of the bar (in addition to the spring and dashpot in parallel), the boundary condition would be

$$AE\frac{\partial u}{\partial x} + m\frac{\partial^2 u}{\partial t^2} + \hat{c}\frac{\partial u}{\partial t} + ku = 0. \tag{2.84}$$

Boundary-initial value problems (BIVPs), which involve time, also have initial conditions specified. Initial conditions are sometimes called Cauchy data.

2.7 Uniqueness

A *well-posed* problem is one for which the solution exists, is unique, and has continuous dependence on the data (geometry, boundary conditions, initial conditions, material properties). Continuous dependence on data means that a small change in the data produces a

small change in the solution. If the solution for some problem exists and is unique, but small changes in data result in big changes in the solution, then the solution is of little value, since all data come from measurements containing small errors.

For transient heat conduction problems, the partial differential equation

$$\nabla^2 T = \frac{1}{K}\frac{\partial T}{\partial t} - \frac{Q}{k} \quad \text{in } V, \tag{2.85}$$

has a unique solution $T(x, y, z, t)$ if there are imposed the boundary condition

$$a\frac{\partial T}{\partial n} + bT = f(x, y, z, t) \quad \text{on } S, \tag{2.86}$$

where a and b are constants, and initial condition

$$T(x, y, z, 0) = T_0(x, y, z). \tag{2.87}$$

The boundary condition should be interpreted as the specification on the boundary of either the temperature T or its gradient $\partial T/\partial n$ or a linear combination of both. If the boundary temperature T is imposed, the boundary condition is known as a Dirichlet condition. If the temperature gradient $\partial T/\partial n$ is imposed, the boundary condition is known as a Neumann condition. The special case $\partial T/\partial n = 0$ is an insulated boundary.

We now prove the above uniqueness claim. If the system of equations 2.85–2.87 had more than one solution $T(x, y, z, t)$, there would be two functions T_1 and T_2, say, both of which satisfy those equations. The difference solution $w = T_2 - T_1$ would therefore be a solution of the corresponding homogeneous system

$$\begin{cases} \nabla^2 w = \dfrac{1}{K}\dfrac{\partial w}{\partial t} \quad \text{in } V, \\ a\dfrac{\partial w}{\partial n} + bw = 0 \quad \text{on } S, \\ w(x, y, z, 0) = 0 \end{cases} \tag{2.88}$$

(A homogeneous equation in w is one for which every term contains the dependent variable w. This use of the term "homogeneous" has nothing to do with its use in describing material properties.) Since the trivial solution $w = 0$ satisfies Eq. 2.88, proving uniqueness for the original problem, Eqs. 2.85–2.87, is equivalent to proving that $w = 0$ is the only solution of the corresponding homogeneous problem, Eq. 2.88.

To verify that $w = 0$ is the only solution of Eq. 2.88, consider the function

$$J(t) = \frac{1}{2K}\int_V w^2 \, dV. \tag{2.89}$$

Since the integrand is non-negative, $J(t) \geq 0$. Also, from the initial condition, Eq. 2.88c, $J(0) = 0$, Then,

$$J'(t) = \frac{1}{K}\int_V w\frac{\partial w}{\partial t} \, dV = \int_V w\nabla^2 w \, dV, \tag{2.90}$$

where the second equation results from the substitution of the original partial differential equation, Eq. 2.88a. Then, from Green's Theorem of the First Kind, Eq. 1.23,

$$J'(t) = \oint_S w \frac{\partial w}{\partial n}\, dS - \int_V \nabla w \cdot \nabla w\, dV = -\left[\oint_S (b/a) w^2\, dS + \int_V \nabla w \cdot \nabla w\, dV \right]. \qquad (2.91)$$

Under the (physically meaningful) assumption that the constants a and b have the same sign, it follows that $J'(t) \leq 0$. That is, $J(t)$ has the characteristics

$$J(0) = 0,\ J(t) \geq 0,\ J'(t) \leq 0, \qquad (2.92)$$

which implies $J(t) \equiv 0$, and, from Eq. 2.89, $w \equiv 0$, and hence $T_1 = T_2$. That is, we have proved that, for the original problem (Eqs. 2.85–2.87), the solution is unique.

A general consequence of the above discussion is that, for linear partial differential equations, a nonhomogeneous system has a unique solution if, and only if, the corresponding homogeneous system has only the trivial solution.

For steady-state (time-independent) heat transfer problems, for which the relevant differential equation is the Poisson or Laplace equation, the general requirement for uniqueness is that a boundary condition of the form Eq. 2.86 be imposed. The only exception is the Neumann problem

$$\begin{cases} \nabla^2 T = f \text{ in } V, \\ \dfrac{\partial T}{\partial n} = g \text{ on } S. \end{cases} \qquad (2.93)$$

Since every appearance of T in this system is in a derivative, solutions of Neumann problems are unique only up to an arbitrary additive constant. Guaranteeing uniqueness in such cases requires that T be specified at one point.

Nonuniqueness in this Neumann problem is analogous to the nonuniqueness which occurs in the static mechanical problem in which a free-free bar is loaded in tension with a force F at each end. Although the stresses in the bar are unique, the displacements are not unique unless the location of one point is prescribed. If the fundamental unknown is the displacement, a force boundary condition is a Neumann condition, since force is proportional to stress, which is proportional to strain, which is proportional to displacement gradient.

2.8 Classification of Partial Differential Equations

Of the classical PDEs derived earlier in this chapter, some involve time, and some don't, so presumably their solutions would exhibit fundamental differences. Of those that involve time (wave and heat equations), the order of the time derivative is different, so the fundamental character of their solutions may also differ. Both these speculations turn out to be true.

Consider the general, second-order, linear partial differential equation in two variables

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \qquad (2.94)$$

where the coefficients are functions of the independent variables x and y (i.e., $A = A(x, y)$, $B = B(x, y)$, etc.), and we have used subscripts to denote partial derivatives. The quantity $B^2 - 4AC$ is referred to as the *discriminant* of the equation. The behavior of the solution of Eq. 2.94 depends on the sign of the discriminant according to the following table:

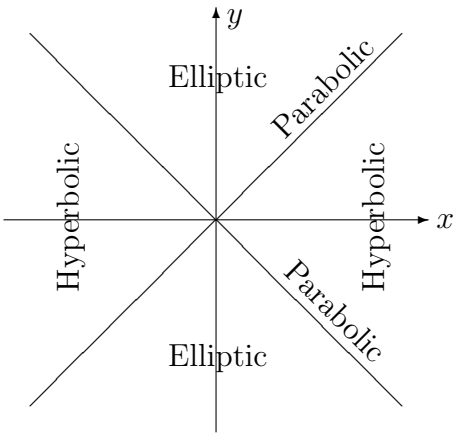


Figure 16: Classification of PDEs with Variable Coefficients.

$B^2 - 4AC$	Equation Type	Typical Physics Described
< 0	Elliptic	Steady-state phenomena
$= 0$	Parabolic	Heat flow and diffusion processes
> 0	Hyperbolic	Vibrating systems and wave motion

The names elliptic, parabolic, and hyperbolic arise from the analogy with the conic sections in analytic geometry.

Given these definitions, we can classify the common equations of mathematical physics already encountered as follows:

Name	Eq. Number	Eq. in Two Variables	A, B, C	Type
Laplace	Eq. 2.34	$u_{xx} + u_{yy} = 0$	$A = C = 1, B = 0$	Elliptic
Poisson	Eq. 2.33	$u_{xx} + u_{yy} = -g$	$A = C = 1, B = 0$	Elliptic
Wave	Eq. 2.14	$u_{xx} - u_{yy}/c^2 = 0$	$A = 1, C = -1/c^2, B = 0$	Hyperbolic
Helmholtz	Eq. 2.18	$u_{xx} + u_{yy} + k^2 u = 0$	$A = C = 1, B = 0$	Elliptic
Heat	Eq. 2.29	$ku_{xx} - \rho c u_y = -Q$	$A = k, B = C = 0$	Parabolic

In the wave and heat equations in the above table, y represents the time variable.

Equations with variable coefficients may be of different types in different domains of the xy-plane. For example, the equation [24]

$$yu_{xx} + 2xu_{xy} + yu_{yy} = 0, \tag{2.95}$$

for which the discriminant is $4(x^2 - y^2)$, is elliptic where $y^2 > x^2$, hyperbolic where $y^2 < x^2$ and parabolic along the lines $y = \pm x$ (Fig. 16).

As will be seen in subsequent sections, the behavior of the solutions of equations of different types will differ. As is already apparent, elliptic equations characterize static (time-independent) situations, and the other two types of equations characterize time-dependent situations. The equations of linear elasticity are hyperbolic for time-dependent problems and elliptic for static and time-harmonic problems. One crutch for helping to remember the classifications is to recall that an ellipse is a closed curve (in analogy with a boundary

value problem), whereas parabolas and hyperbolas are open curves (in analogy with time-dependent problems with no limit on time.) Note also that a parabola, like the heat equation in 1-D, is second order in one variable and first order in the other variable.

The Laplace, Poisson, wave, Helmholtz, and heat equations are classical in mathematical physics and arise in many physical situations. For example, Laplace's equation (the potential equation) arises in incompressible fluid flow (where the fundamental unknown is the velocity potential), gravitational potential problems, electrostatics, magnetostatics, and steady-state heat conduction. The Poisson equation arises in steady-state heat conduction (with distributed sources) and torsion of prismatic bars in elasticity. The wave and Helmholtz equations arise, respectively, in transient and time-harmonic elastic vibrations (strings, bars, and membranes), acoustics, and electromagnetics. The heat equation arises in heat conduction and other diffusion processes.

2.9 Nature of Solutions of PDEs

To illustrate the differences between hyperbolic and parabolic problems, consider the two well-posed problems defined in the following table:

	Hyperbolic	Parabolic
Physics	String released from rest	Cooling of heated rod
Dependent Variable	Transverse displacement $u(x,t)$	Temperature $u(x,t)$
PDE	$c^2 u_{xx} = u_{tt}$ (wave equation)	$K u_{xx} = u_t$ (heat equation)
B.C.	$u(0,t) = u(L,t) = 0$	$u(0,t) = u(L,t) = 0$
I.C.	$u(x,0) = \sin(\pi x/L)$, $u_t(x,0) = 0$	$u(x,0) = \sin(\pi x/L)$
Solution	$u(x,t) = \sin(\pi x/L)\cos(\pi ct/L)$	$u(x,t) = \sin(\pi x/L)e^{-K(\pi/L)^2 t}$

The hyperbolic problem describes a string of length L released from rest from the initial shape $\sin(\pi x/L)$. The parabolic problem describes a rod of length L cooled from the initial temperature distribution $\sin(\pi x/L)$. The only difference between the wave and heat equations above is the order of the time derivatives. Both problems have the same zero boundary conditions. The solutions can be verified by direct substitution into the PDEs, B.C., and I.C. Both solutions have the same shape, $\sin(\pi x/L)$, but the string oscillates sinusoidally with period $2L/c$ [or frequency c/(2L)], whereas the temperature distribution in the cooling problem decays exponentially without oscillation.

The above heat solution also shows that the thermal diffusivity $K = k/(\rho c)$ describes the rate at which heat is conducted through a medium. Materials with high thermal diffusivity conduct heat quickly.

The wave equation, as the name implies, characterizes wave propagation. For example, the time-dependent transverse response of an infinitely long string satisfies the one-dimensional wave equation with nonzero initial displacement and velocity specified:

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x) \\ &\lim_{x \to \pm\infty} u(x,t) = 0, \end{aligned} \tag{2.96}$$

where x is distance along the string, t is time, $u(x,t)$ is the transverse displacement, $f(x)$ is the initial displacement, $g(x)$ is the initial velocity, and the constant c is given by

$$c = \sqrt{\frac{T}{\rho A}}, \tag{2.97}$$

where T is the tension (force) in the string, ρ is the density of the string material, and A is the cross-sectional area of the string. Note that c has the dimension of velocity. Recall that this equation assumes that all motion is vertical and that the displacement u and its slope $\partial u/\partial x$ are both small.

It can be shown by direct substitution into Eq. 2.96 that the solution of this system is

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau. \tag{2.98}$$

The differentiation of the integral in Eq. 2.98 is effected with the aid of Leibnitz's rule:

$$\frac{d}{dx}\int_{A(x)}^{B(x)} h(x,t)\,dt = \int_A^B \frac{\partial h(x,t)}{\partial x}\,dt + h(x,B)\frac{dB}{dx} - h(x,A)\frac{dA}{dx}. \tag{2.99}$$

Eq. 2.98 is known as the d'Alembert solution of the one-dimensional wave equation.

For the special case $g(x) = 0$ (zero initial velocity), the d'Alembert solution simplifies to

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)], \tag{2.100}$$

which may be interpreted as two waves, each equal to $f(x)/2$, which travel at speed c to the right and left, respectively. For example, the argument $x - ct$ remains constant if, as t increases, x also increases at speed c. Thus, the wave $f(x-ct)$ moves to the right (increasing x) with speed c without change of shape. Similarly, the wave $f(x+ct)$ moves to the left (decreasing x) with speed c without change of shape. The two waves, each equal to half the initial shape $f(x)$, travel in opposite directions from each other at speed c. If $f(x)$ is nonzero only for a small domain, then, after both waves have passed the region of initial disturbance, the string returns to its rest position.

For example, let $f(x)$, the initial displacement, be given by

$$f(x) = \begin{cases} -|x| + b, & |x| \le b, \\ 0, & |x| \ge b, \end{cases} \tag{2.101}$$

which is a triangular pulse of width $2b$ and height b (Fig. 17). For $t > 0$, half this pulse travels in opposite directions from the origin. For $t > b/c$, where c is the wave speed, the two half-pulses have completely separated, and the neighborhood of the origin has returned to rest.

For the special case $f(x) = 0$ (zero initial displacement), the d'Alembert solution simplifies to

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau = \frac{1}{2}\left[G(x+ct) - G(x-ct)\right], \tag{2.102}$$

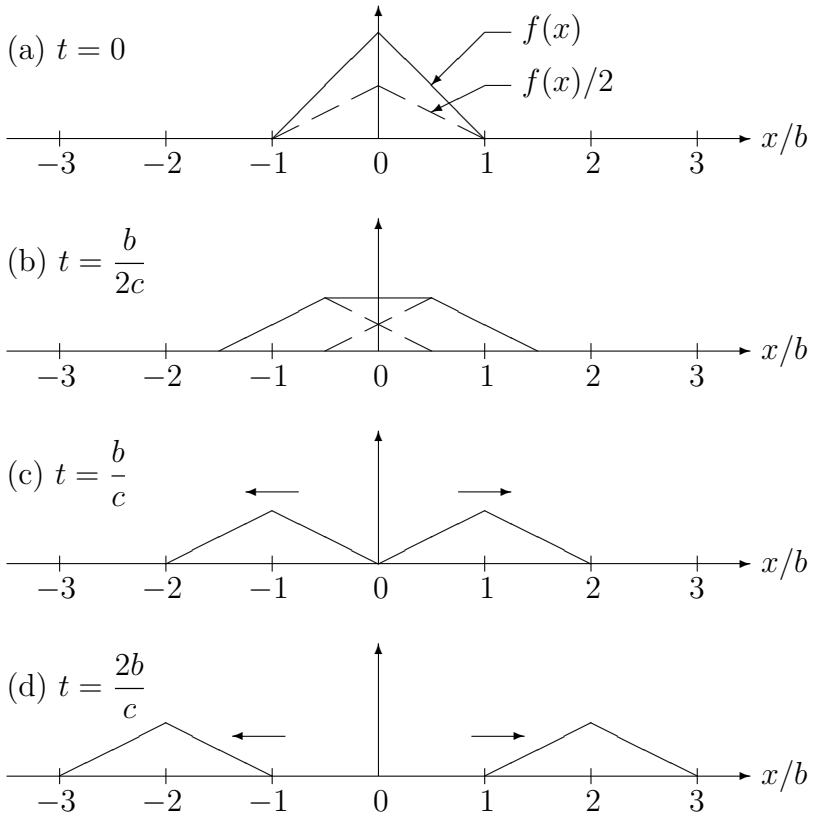


Figure 17: Propagation of Initial Displacement.

where

$$G(x) = \frac{1}{c}\int_{-\infty}^{x} g(\tau)\, d\tau. \tag{2.103}$$

Thus, similar to the initial displacement special case, this solution, Eq. 2.102, may be interpreted as the combination (difference, in this case) of two identical functions $G(x)/2$, one moving left and one moving right, each with speed c.

For example, let the initial velocity $g(x)$ be given by

$$g(x) = \begin{cases} Mc, & |x| < b, \\ 0, & |x| > b, \end{cases} \tag{2.104}$$

which is a rectangular pulse of width $2b$ and height Mc, where the constant M is the dimensionless Mach number, and c is the wave speed (Fig. 18). The travelling wave $G(x)$ is given by Eq. 2.103 as

$$G(x) = \begin{cases} 0, & x \le -b, \\ M(x+b), & -b \le x \le b, \\ 2Mb, & x \ge b. \end{cases} \tag{2.105}$$

That is, half this wave travels in opposite directions at speed c. Even though $g(x)$ is nonzero only near the origin, the travelling wave $G(x)$ is constant and nonzero for $x > b$. Thus, as

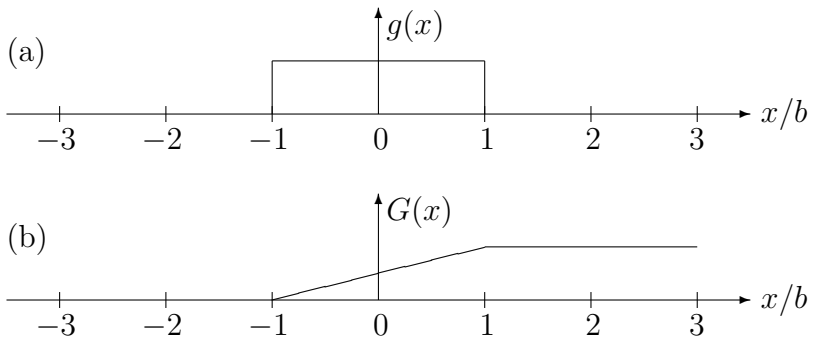


Figure 18: Initial Velocity Function.

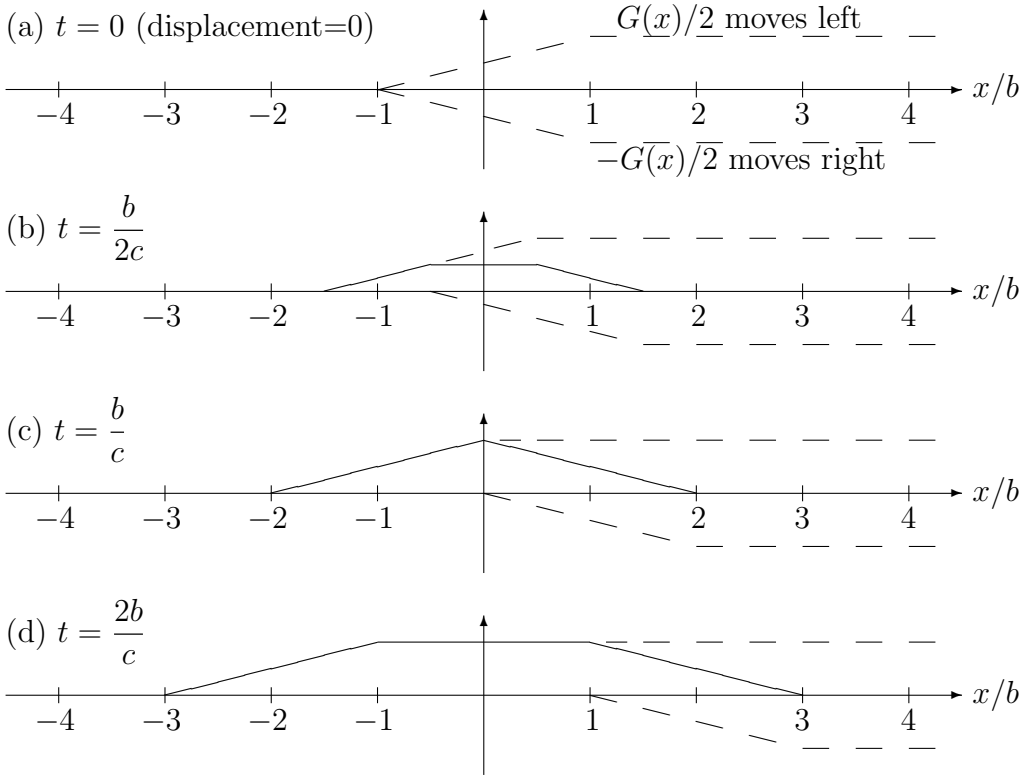
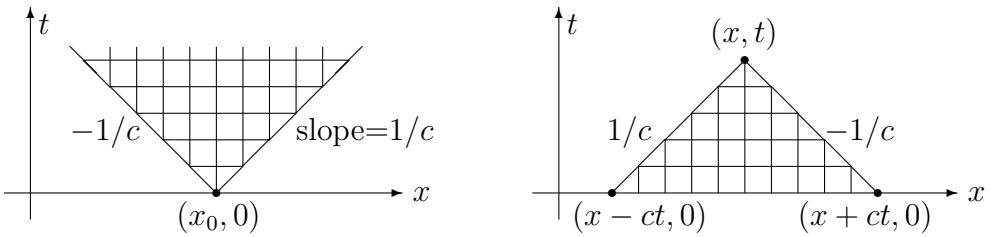


Figure 19: Propagation of Initial Velocity.



(a) Domain of Influence. (b) Domain of Dependence.

Figure 20: Domains of Influence and Dependence.

time advances, the center section of the string reaches a state of rest, but not in its original position (Fig. 19).

From the preceding discussion, it is clear that disturbances travel with speed c. For an observer at some fixed location, initial displacements occurring elsewhere pass by after a finite time has elapsed, and then the string returns to rest in its original position. Nonzero initial velocity disturbances also travel at speed c, but, once having reached some location, will continue to influence the solution from then on.

Thus, the *domain of influence* of the data at $x = x_0$, say, on the solution consists of all points closer than ct (in either direction) to x_0, the location of the disturbance (Fig. 20). Conversely, the *domain of dependence* of the solution on the initial data consists of all points within a distance ct of the solution point. That is, the solution at (x, t) depends on the initial data for all locations in the range $(x - ct, x + ct)$, which are the limits of integration in the d'Alembert solution, Eq. 2.98.

A few examples of propagation speeds are listed in the following table:

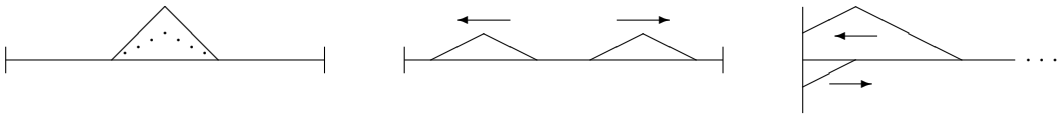
Material	Propagation Speed c
Steel	5000 m/s
Aluminum	5100 m/s
Water	1485 m/s
Seawater	1545 m/s
Air	340 m/s

For solids, the speed is given by Eq. 2.21.

The d'Alembert solution can also be adapted to handle the reflection of waves at the end supports of finite length strings. Consider again the initial disturbance of Fig. 17. If the string occupies the domain $0 \leq x \leq L$, waves impinging upon the walls will reflect. Since the nature of the reflection is the same at both ends, we can consider the end $x = 0$, where the boundary condition requires

$$u(0, t) = \frac{1}{2}[f(-ct) + f(ct)] = 0. \tag{2.106}$$

Thus, any portion of the wave that "passes through" the wall is reflected back into the domain $0 \leq x \leq L$ in such a way that $f(x) = -f(-x)$, as illustrated in Fig. 21. That is, a wave hitting a fixed end "flips over." In Fig. 21c, the two sections of wave are added to yield



(a) Initial Displacement (b) A Little Later (c) "Entering" Wall

Figure 21: Reflection at a Wall.

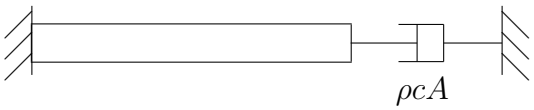


Figure 22: The Nonreflecting Boundary Condition for Longitudinal Waves in a Bar.

the resulting wave at that time. Since the wave flips over every time it hits a wall, we can also conclude that the string will recover its original shape after one full round trip involving a distance $2L$. Since the wave propagation occurs at speed c, the period of the oscillation is therefore $2L/c$, and the frequency, which is the inverse of the period, is $c/(2L)$.

The d'Alembert solution can be used to determine the non-reflecting boundary condition for 1-D wave motion. That is, if a finite length bar on the x-axis has a longitudinal wave propagating to the right at speed c, we wish to determine the appropriate boundary condition to absorb all such waves so that there would be no reflection at the end. From the d'Alembert solution of the wave equation, the wave travelling to the right in the bar would be of the form $u(x - ct)$. We differentiate u with respect to x and t to obtain

$$\frac{\partial u}{\partial x} = u', \quad \frac{\partial u}{\partial t} = -cu', \tag{2.107}$$

where a prime denotes the derivative with respect to the argument. The elimination of u' from these two equations yields

$$\frac{\partial u}{\partial x} + \frac{1}{c}\frac{\partial u}{\partial t} = 0, \tag{2.108}$$

which is the desired one-dimensional nonreflecting boundary condition. A comparison of this equation with Eq. 2.82 shows that Eq. 2.108 represents a damper whose damping coefficient $\hat{c}$ is given by

$$\hat{c} = \frac{AE}{c} = \rho c A, \tag{2.109}$$

where Eq. 2.21 was used. In this equation, ρ is the density of the bar material, c is the propagation speed in the bar (the bar speed), and A is the cross-sectional area of the bar. Thus, if a bar is terminated with a dashpot of that size, as shown in Fig. 22, all waves impinging upon the end of the bar will be absorbed with no reflection. This condition is exact in 1-D, where the waves are planar. If a bar is terminated with any impedance other than that shown in Fig. 22, some portion (or all) of the energy of a wave impinging upon the boundary will reflect.

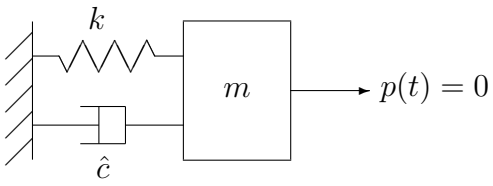


Figure 23: Single Degree of Freedom Mechanical System.

Now consider the one-dimensional *damped* wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{1}{K}\frac{\partial u}{\partial t}, \tag{2.110}$$

which is hyperbolic. The first derivative term can be seen to be a damping term by drawing the analogy between this equation and the single degree of freedom (SDOF) mass-spring-dashpot equation

$$m\ddot{u} + \hat{c}\dot{u} + ku = p(t) = 0, \tag{2.111}$$

which is the equation of motion for the mechanical system shown in Fig. 23. However, the PDE is also a composite equation, since the hyperbolic equation reduces to the parabolic heat equation if $c = \infty$. Thus, by an informal argument, we conclude that disturbances travel with infinite speed of propagation when modeled by the heat equation. We will see later in the examples that local disturbances are indeed felt elsewhere immediately, but that a very sensitive instrument would be required to detect the disturbance. Since, physically, nothing can propagate at infinite speed, this property of the heat equation can be viewed as an artifact of the model. For most practical purposes, however, the artifact is insignificant.

2.10 Transformation to Nondimensional Form

It is sometimes convenient, when solving equations, to transform the equation to a nondimensional form. Consider, for example, the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{2.112}$$

where u is displacement (of dimension length), t is time, and c is the speed of propagation (of dimension length/time). Let L represent some characteristic length associated with the problem. We define the nondimensional variables

$$\bar{x} = \frac{x}{L}, \quad \bar{u} = \frac{u}{L}, \quad \bar{t} = \frac{ct}{L}. \tag{2.113}$$

If we substitute Eq. 2.113 into Eq. 2.112, and factor out the constants, we obtain

$$\frac{\partial^2 (L\bar{u})}{\partial (L\bar{x})^2} = \frac{1}{c^2}\frac{\partial^2 (L\bar{u})}{\partial (L\bar{t}/c)^2} \tag{2.114}$$

or

$$\frac{L}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = \frac{1}{c^2}\frac{c^2 L}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} \tag{2.115}$$

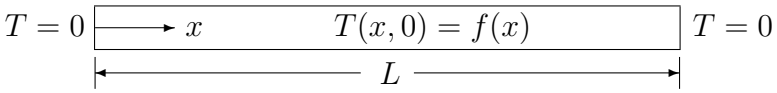


Figure 24: 1-D Heat Conduction in a Rod.

or

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = \frac{\partial^2 \bar{u}}{\partial \bar{t}^2}. \tag{2.116}$$

This is the nondimensional wave equation.

3 Fourier Series Solutions

3.1 A One-Dimensional Heat Conduction Problem

Consider the transient heat conduction problem of a bar with a prescribed initial temperature distribution, no internal heat sources, and zero temperature prescribed at both ends (Fig. 24). It is desired to determine the temperature solution $T(x,t)$. The mathematical statement of this problem is

$$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{K}\dfrac{\partial T}{\partial t}, \quad 0 < x < L, \\ T(0,t) = T(L,t) = 0, \\ T(x,0) = f(x), \end{cases} \tag{3.1}$$

where L is the length of the bar, and $f(x)$ is the initial distribution of temperature prescribed over the length of the bar. The problem is to find $T(x,t)$ for $t > 0$. Given the discussion in §2.7, we note that this is a well-posed problem.

We look for a solution of Eq. 3.1 using the technique of separation of variables:

$$T(x,t) = X(x)\mathcal{T}(t). \tag{3.2}$$

The substitution of Eq. 3.2 into Eq. 3.1a leads to

$$X''\mathcal{T} = \frac{1}{K} X \mathcal{T}' \tag{3.3}$$

(where primes denote derivatives) or

$$\frac{X''}{X} = \frac{1}{K}\frac{\mathcal{T}'}{\mathcal{T}}, \tag{3.4}$$

implying that, for all x and t, a function of x is equal to a function of t. Thus, each function must be constant:

$$\frac{X''}{X} = \frac{1}{K}\frac{\mathcal{T}'}{\mathcal{T}} = a, \tag{3.5}$$

or

$$\begin{cases} X'' - aX = 0, \\ \mathcal{T}' - Ka\mathcal{T} = 0. \end{cases} \tag{3.6}$$

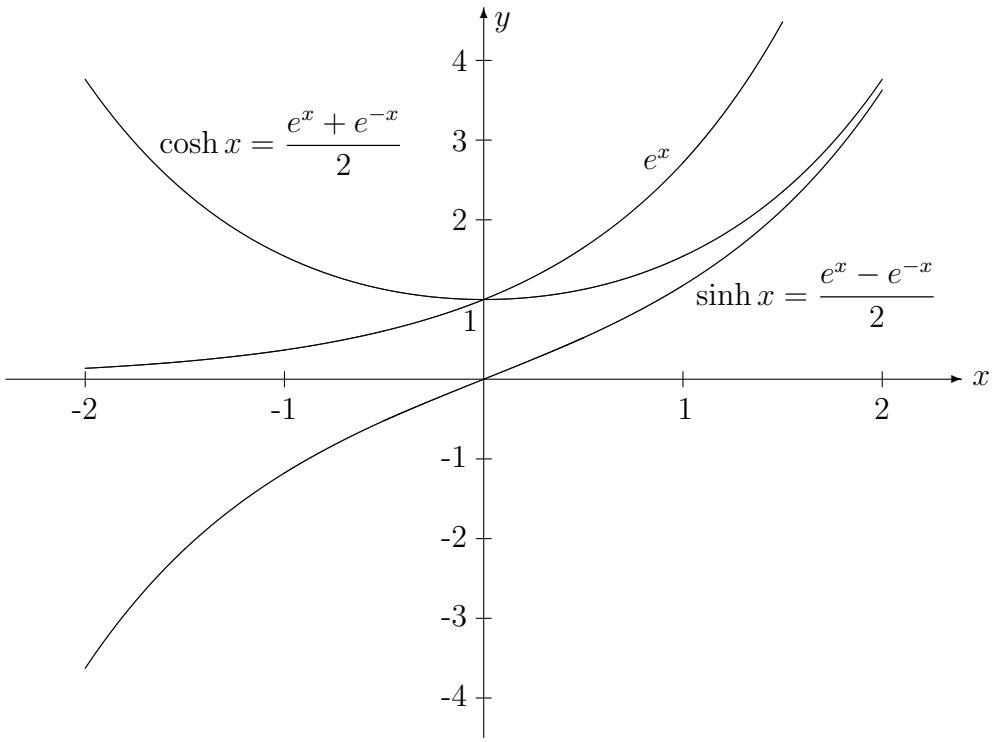


Figure 25: Hyperbolic Functions.

Since the separation constant a can be positive, zero, or negative, we consider each case in turn.

Case 1: $a > 0$. Under the assumption that $a > 0$, the general solution of Eq. 3.6a is

$$X(x) = B \sinh \sqrt{a}\, x + C \cosh \sqrt{a}\, x, \tag{3.7}$$

where B and C are constants. The boundary conditions, Eq. 3.1b, imply that, for all t,

$$T(0,t) = X(0)\mathcal{T}(t) = 0, \quad T(L,t) = X(L)\mathcal{T}(t) = 0 \tag{3.8}$$

and, hence, $X(0) = X(L) = 0$. Thus, from Eq. 3.7, $X(x) \equiv 0$, and $T(x,t) \equiv 0$. Since this function does not satisfy the initial condition, we conclude that the separation constant a cannot be positive.

The hyperbolic functions cosh and sinh, shown in Fig. 25, have many properties analogous to the circular functions cos and sin. The circular functions parameterize the circle, since, with $x = \cos u$ and $y = \sin u$, we have $x^2 + y^2 = 1$, which is the unit circle. On the other hand, cosh and sinh parameterize the hyperbola, since, with $x = \cosh u$ and $y = \sinh u$, we have $x^2 - y^2 = 1$, which is the "unit hyperbola." (In parametric form, this is the hyperbola located in the right-half plane, symmetrical with respect to the x-axis, and asymptotic to the two lines $y = \pm x$.)

Case 2: $a = 0$. The second possibility, $a = 0$, cannot be considered a special case of Case 1, since, with $a = 0$, the two independent solutions in Eq. 3.7 reduce to only one solution. Thus, for this case, we must return to the original ordinary differential equation, Eq. 3.6a:

$$X''(x) = 0, \; X(0) = X(L) = 0, \tag{3.9}$$

the solution of which is $X(x) \equiv 0$. Thus, a cannot be zero.

Case 3: $a < 0$. We now consider the third possibility, $a < 0$. For convenience, we define $a = -\alpha^2$, and Eq. 3.6 becomes

$$\begin{cases} X'' + \alpha^2 X = 0, \\ \mathcal{T}' + K\alpha^2 \mathcal{T} = 0, \end{cases} \tag{3.10}$$

whose general solutions are

$$\begin{cases} X(x) = B \sin \alpha x + C \cos \alpha x, \\ \mathcal{T}(t) = A e^{-K\alpha^2 t}. \end{cases} \tag{3.11}$$

The boundary condition $X(0) = 0$ implies $C = 0$. The boundary condition $X(L) = 0$ implies $B \sin \alpha L = 0$, or $\alpha L = n\pi$ for n any positive integer. Hence,

$$\alpha = \frac{n\pi}{L}, \ n = 1, 2, 3, \ldots \tag{3.12}$$

The combination of the X and $\mathcal{T}$ solutions yields

$$T(x,t) = A \sin \frac{n\pi x}{L} e^{-K(n\pi/L)^2 t}, \tag{3.13}$$

which satisfies the partial differential equation and the boundary conditions. The one remaining condition, the initial condition, is not satisfied by this function, since, for arbitrary $f(x)$,

$$f(x) \neq A \sin \frac{n\pi x}{L}. \tag{3.14}$$

However, since $T(x,t)$ given by Eq. 3.13 is a solution for any positive integer n, we can add functions of that form to get more solutions. Thus, the most general solution of the homogeneous part of the original system, Eq. 3.1, is

$$T(x,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} e^{-K(n\pi/L)^2 t}. \tag{3.15}$$

Like Eq. 3.13, this function satisfies the partial differential equation and the boundary conditions.

To satisfy the initial condition, we require

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}, \tag{3.16}$$

where the coefficients A_n are unknown. A convenient way to evaluate A_n is provided by the relation

$$\int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \begin{cases} 0, & m \neq n \\ L/2, & m = n, \end{cases} \tag{3.17}$$

where m and n are integers. If we then multiply both sides of Eq. 3.16 by $\sin(m\pi x/L)$ and integrate, we obtain

$$\int_0^L f(x) \sin \frac{m\pi x}{L} \, dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = A_m \frac{L}{2}, \tag{3.18}$$

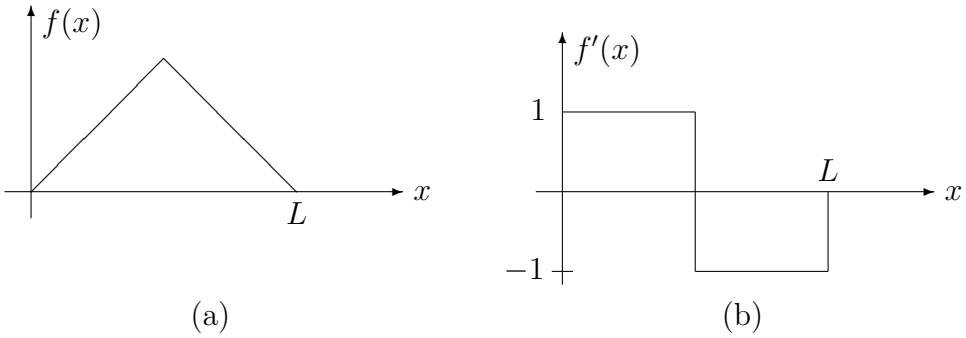


Figure 26: Example of Initial Temperature Distribution.

where we have made the assumption that we can interchange the order of summation and integration. (That is, we have assumed that, for an infinite series, the integral of a sum is the sum of the integrals.) Thus, from Eq. 3.18,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{3.19}$$

Therefore, the solution of the original problem, Eq. 3.1, is the infinite series, Eq. 3.15, with the coefficients given by Eq. 3.19. Since the original problem is a well-posed problem, and our solution satisfies the partial differential equation, the boundary data, and the initial condition, we have found the unique solution.

Note that simple jump discontinuities are allowed in the initial temperature function $f(x)$, since the only place that f appears in the solution is in an integral (Eq. 3.19). Jump discontinuities in f have no effect on the integral.

The problem solved in this section has one forcing function (a nonhomogeneous term in the PDE, boundary conditions, or initial condition). That forcing function is the initial temperature distribution $f(x)$. The solution approach involved decomposing the general forcing function $f(x)$ into an infinite series of sine functions, each one of which yields one term of the solution. Thus we have solved the problem by superposing solutions for each harmonic of the decomposition. Superposition works only when the system is linear.

As we will see in subsequent sections, the series representation of the general function $f(x)$ in Eq. 3.16 with coefficients given by Eq. 3.19 is referred to as the Fourier sine series representation of $f(x)$. Note that the terms in the series in Eq. 3.15 get small rapidly as n increases. Thus, to compute $T(x,t)$ numerically, one can include as many terms as are needed for convergence of the series.

3.2 Example 1

To illustrate the representation of an arbitrary function $f(x)$ with the series, Eq. 3.16, let

$$f(x) = \begin{cases} x, & 0 \le x \le L/2, \\ L - x, & L/2 \le x \le L. \end{cases} \tag{3.20}$$

This function and its first derivative are shown in Fig. 26. From Eq. 3.19,

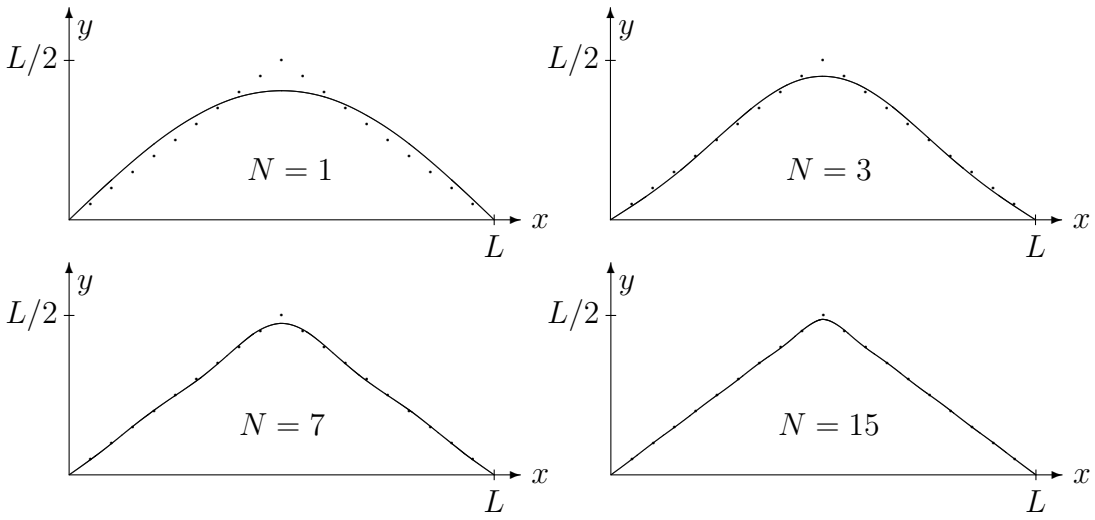


Figure 27: Convergence of Series for Example 1.

$$A_n = \frac{2}{L}\int_0^{L/2} x \sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{4L}{(n\pi)^2}\sin\frac{n\pi}{2} = \begin{cases} \dfrac{4L}{(n\pi)^2}(-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \tag{3.21}$$

Retaining only the odd terms makes sense, since $f(x)$ and the sine functions for n odd are symmetrical with respect to the mid-point $x = L/2$. The sine functions for n even are antisymmetrical with respect to $x = L/2$.

The convergence of the series, Eq. 3.16, can be seen by evaluating the partial sum

$$f(x) \approx \sum_{n=1}^{N} A_n \sin\frac{n\pi x}{L} \tag{3.22}$$

for different values of the upper limit N. Four such representations of $f(x)$ are shown in Fig. 27 for $N = 1$, 3, 7, and 15. Since only the odd terms are nonzero for this series, the four partial sums have 1, 2, 4, and 8 nonzero terms, respectively.

An alternative way to evaluate the integral, Eq. 3.19, for this $f(x)$ is provided by the Dirac delta function defined in §1.4. From Eq. 3.20,

$$f'(x) = \begin{cases} 1, & 0 < x < L/2, \\ -1, & L/2 < x < L, \end{cases} \tag{3.23}$$

from which it follows that

$$f''(x) = -2\delta(x - L/2). \tag{3.24}$$

Also, from the series, Eq. 3.16, term-by-term differentiation yields

$$f''(x) = -\sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi x}{L}. \tag{3.25}$$

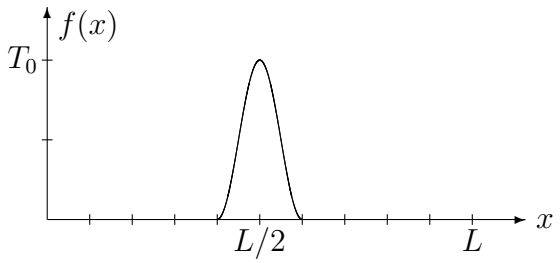


Figure 28: Localized Initial Temperature Distribution.

If we now multiply both sides of this equation by $\sin(m\pi x/L)$ and integrate, we obtain

$$\int_0^L f''(x) \sin \frac{n\pi x}{L}\, dx = -A_n \left(\frac{n\pi}{L}\right)^2 \frac{L}{2}. \tag{3.26}$$

If Eq. 3.24 is then substituted into Eq. 3.26, the previous result, Eq. 3.21, is obtained.

3.3 Example 2

As a second example of the series, Eq. 3.16, let

$$f(x) = \begin{cases} 0, & 0 \le x \le (1-\alpha)L/2, \\ \dfrac{T_0}{2}\left[1 + \cos\dfrac{2\pi}{\alpha L}\left(x - \dfrac{L}{2}\right)\right], & (1-\alpha)L/2 \le x \le (1+\alpha)L/2, \\ 0, & (1+\alpha)L/2 \le x \le L, \end{cases} \tag{3.27}$$

where $\alpha = 0.2$ (Fig. 28). The coefficients of the sine series representation are given by Eq. 3.19 as

$$\begin{aligned} A_n &= \frac{2}{L}\int_{(1-\alpha)L/2}^{(1+\alpha)L/2} \frac{T_0}{2}\left[1 + \cos\frac{2\pi}{\alpha L}\left(x - \frac{L}{2}\right)\right] \sin\frac{n\pi x}{L}\, dx \\ &= \begin{cases} \dfrac{8T_0}{n\pi(4 - n^2\alpha^2)} \sin\dfrac{n\pi}{2} \sin\dfrac{n\pi\alpha}{2}, & n \ne 2/\alpha, \\ \dfrac{T_0\alpha}{2}\sin\dfrac{n\pi}{2}, & n = 2/\alpha, \end{cases} \end{aligned} \tag{3.28}$$

where α is fixed. The special case $n\alpha = 2$ can be derived using l'Hôpital's rule. As expected from the symmetry of $f(x)$, $A_n = 0$ for n even.

The convergence of the series, Eq. 3.16, as a function of the number N of terms used is shown in Fig. 29. Summing 31 terms of the series (half of which are zero) yields a function indistinguishable from $f(x)$. It is also interesting to plot the time history of the temperature at $x/L = 0.2$ and $x/L = 0.5$ (Fig. 30). Nondimensional time $\hat{t} = K(\pi/L)^2 t$ is displayed in that figure. Notice that the temperature at $x/L = 0.2$ starts to rise without delay at $t = 0$, as shown more clearly in the early-time enlargement in Fig. 31. In effect, according to the heat equation, local disturbances travel with infinite speed, although the changes in temperature at $x/L = 0.2$ for very early time may be too small to detect.

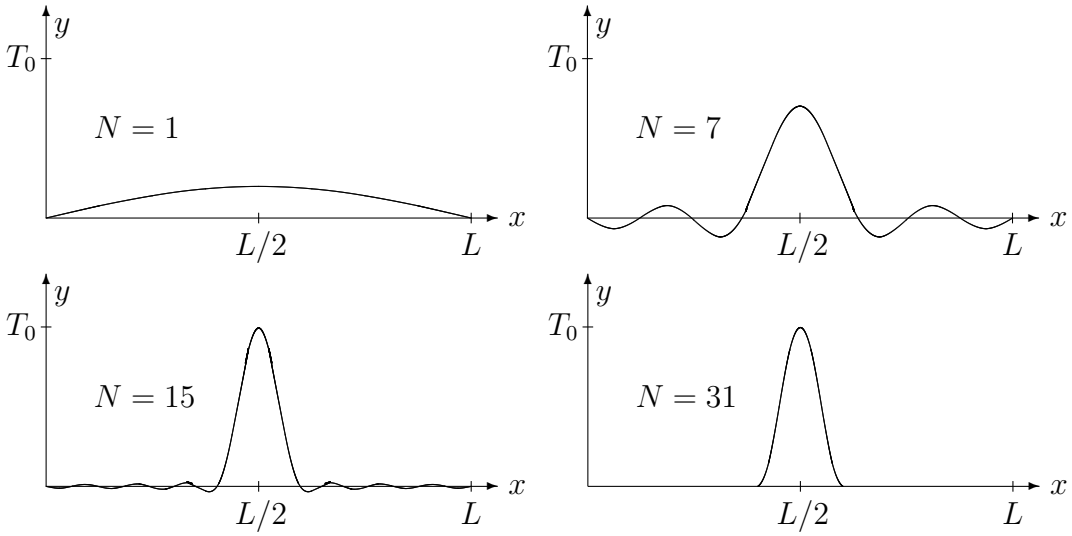


Figure 29: Convergence of Series for Example 2.

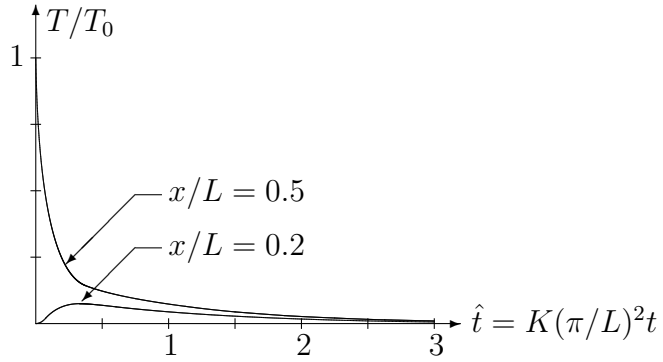


Figure 30: Temperature History at Two Locations.

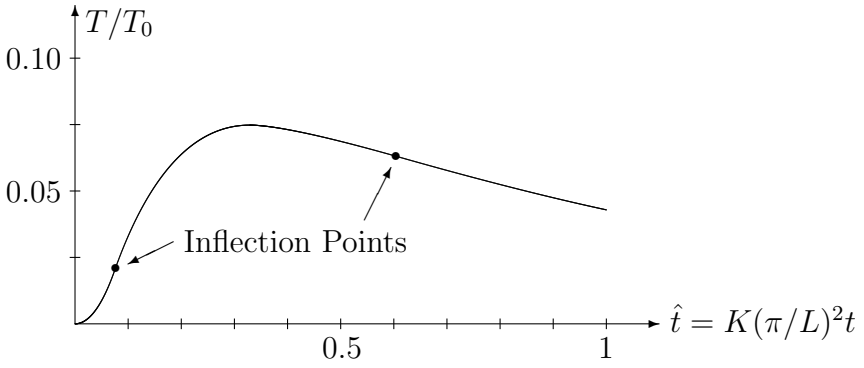


Figure 31: Early Temperature History at $x/L = 0.2$.

3.4 Alternatives for Coefficient Evaluation

Previously, when faced with the need to evaluate the coeffficients A_n in

$$f(x) \approx \sum_{n=1}^{N} A_n \sin \frac{n\pi x}{L}, \tag{3.29}$$

we found it convenient to multiply both sides of the equation by $\sin(m\pi x/L)$ and integrate to obtain

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{3.30}$$

The equality in Eq. 3.29 is approximate, since a finite number N of terms cannot represent exactly an arbitrary function. We denote this approach Method 1. Our interest here is whether there are other methods to evaluate A_n and, if so, the reasons for preferring one method over another.

Method 2. A second method to evaluate A_n can be developed by multiplying both sides of Eq. 3.29 by powers of x rather than trigonometric functions. For example, if we multiply Eq. 3.29 by x and integrate, we obtain

$$\int_0^L x f(x)\, dx \approx \sum_{n=1}^{N} A_n \int_0^L x \sin \frac{n\pi x}{L}\, dx. \tag{3.31}$$

Similarly, multiplying by x^2 yields

$$\int_0^L x^2 f(x)\, dx \approx \sum_{n=1}^{N} A_n \int_0^L x^2 \sin \frac{n\pi x}{L}\, dx. \tag{3.32}$$

These two equations are the first two of a system of N linear algebraic equations

$$\left\{ \begin{array}{rcl} a_{11}A_1 + a_{12}A_2 + \cdots + a_{1N}A_N &=& f_1 \\ a_{21}A_1 + a_{22}A_2 + \cdots + a_{2N}A_N &=& f_2 \\ \vdots & & \\ a_{N1}A_1 + a_{N2}A_2 + \cdots + a_{NN}A_N &=& f_N, \end{array} \right. \tag{3.33}$$

where the unknowns are A_1, A_2, ..., A_N, and

$$a_{ij} = \int_0^L x^i \sin \frac{j\pi x}{L}\, dx, \quad f_i = \int_0^L x^i f(x)\, dx. \tag{3.34}$$

Since this system of equations is coupled (i.e., it is a non-diagonal system), it can be solved numerically using standard equation solvers. Notice that, if the number N of terms is increased, the system must be formed and solved again for all unknowns.

Method 3. A third method of evaluating the coefficients A_n is referred to as *collocation*. The summation in Eq. 3.29 can be forced to match the function $f(x)$ at exactly N points (Fig. 32), thus yielding N equations in the same N unknowns.

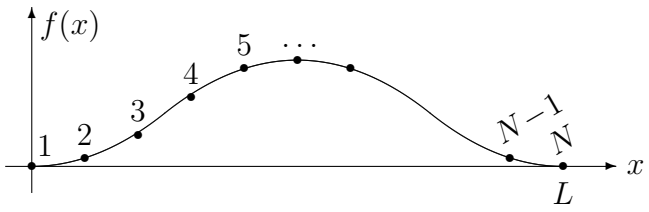


Figure 32: Collocation Points.

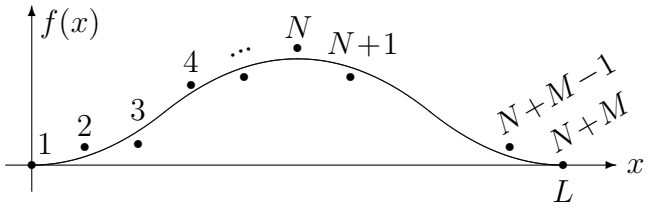


Figure 33: Least Squares Fit.

Method 4. We can improve on Method 3 by attempting to force the summation to go through more than N points, say $N+M$ points (Fig. 33). Since there are only N unknown coefficients but $N+M$ equations, a least squares fit of the summation to $f(x)$ could be used. In the least squares algorithm, the unknown coefficients A_n are picked so as to minimize the mean square error between the function and the series approximation at the $N+M$ points. Thus, we want to minimize

$$\bar{M} = \sum_{p=1}^{N+M} \left[f(x_p) - \sum_{n=1}^{N} A_n \sin \frac{n\pi x_p}{L} \right]^2 . \tag{3.35}$$

(Since $\bar{M}$ is the sum of the squares of the errors, minimizing $\bar{M}$ is equivalent to minimizing the mean square error.) To minimize $\bar{M}$, we set

$$\frac{\partial \bar{M}}{\partial A_i} = 0, \ i = 1, 2, \cdots, N \tag{3.36}$$

Method 5. A fifth method can be developed by improving Method 4 so that we minimize the error *continuously* between $x = 0$ and $x = L$. We thus want to pick the N values of A_n so as to minimize

$$M = \int_0^L \left[f(x) - \sum_{n=1}^{N} A_n \sin \frac{n\pi x}{L} \right]^2 dx. \tag{3.37}$$

In this case, we set

$$\frac{\partial M}{\partial A_i} = 0, \ i = 1, 2, \cdots, N, \tag{3.38}$$

yielding

$$0 = \int_0^L 2 \left[f(x) - \sum_{n=1}^{N} A_n \sin \frac{n\pi x}{L} \right] \left(- \sin \frac{i\pi x}{L} \right) dx \tag{3.39}$$

or

$$\int_0^L f(x) \sin \frac{i\pi x}{L}\, dx = \sum_{n=1}^{N} A_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{i\pi x}{L}\, dx = A_i \frac{L}{2}, \tag{3.40}$$

where we have interchanged the order of integration and summation. Thus,

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx. \tag{3.41}$$

That is, if we minimize the mean square error between the function and the series approximation, we obtain the same result as in Method 1, Eq. 3.30. Method 1 and Method 5 are, in fact, the same method.

In Methods 1 and 5, once a coefficient A_n has been evaluated, it does not change or have to be re-calculated if the number N of terms changes. In the other three methods, if we change N (to improve the approximation, for example), we must re-calculate all coefficients, not just the new ones.

The coefficients A_n in Eq. 3.29 are referred to as the *Fourier coefficients* of $f(x)$, and the series is the *Fourier sine series* corresponding to $f(x)$. (There is also a cosine series, as we will see in the next section.) In general, a Fourier series is a series representation of a function which minimizes the mean square error between the series and the function along the length of the interval. We assume that minimizing the mean square error is desirable.

3.5 Complete Fourier Series

As we saw in §3.1, any reasonably well-behaved function $f(x)$ can be represented in the interval $(0, L)$ by a series consisting of sines:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}. \tag{3.42}$$

Similarly, $f(x)$ can be represented over $(0, L)$ by a series of cosines:

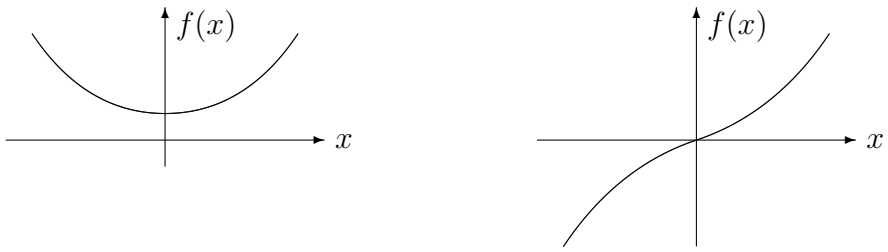
$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \tag{3.43}$$

where A_n can be determined similar to the way B_n was determined for the sine series. We multiply both sides of Eq. 3.43 by $\cos(m\pi x/L)$ and integrate to obtain

$$\int_0^L f(x) \cos \frac{m\pi x}{L}\, dx = A_0 \int_0^L \cos \frac{m\pi x}{L}\, dx + \sum_{n=1}^{\infty} A_n \int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}\, dx, \tag{3.44}$$

where we have interchanged the order of integration and summation. The integrals on the right-hand side are

$$\int_0^L \cos \frac{m\pi x}{L}\, dx = \begin{cases} 0, & m \neq 0, \\ L, & m = 0, \end{cases} \tag{3.45}$$



(a) Even (b) Odd

Figure 34: Examples of Even and Odd Functions.

and

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0, & m \neq n, \\ L/2, & m = n, \end{cases} \tag{3.46}$$

where m and n are integers. Thus,

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \tag{3.47}$$

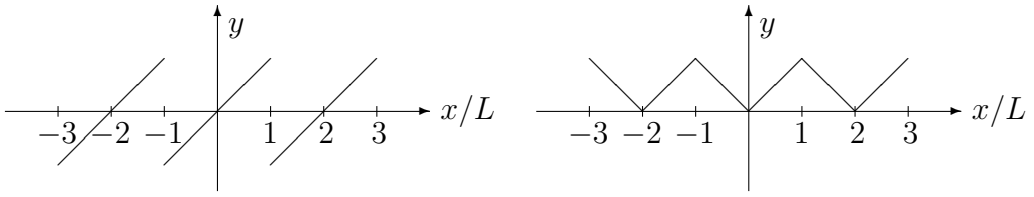
$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\ n = 1, 2, 3, \cdots\ . \tag{3.48}$$

We conclude that any function $f(x)$ can be represented in $(0, L)$ by either a sine series or a cosine series.

Before generalizing the above development to the interval $(-L, L)$, we first define some terms. A function $f(x)$ is *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$, as illustrated in Fig. 34. Geometrically, an even function is *symmetrical* with respect to the line $x = 0$, and an odd function is *antisymmetrical*. For example, $\cos x$ is even, and $\sin x$ is odd.

Even and odd functions have the following useful properties:

1. If f is even and *smooth* (continuous first derivative), $f'(0) = 0$.
2. If f is odd, $f(0) = 0$.
3. The product of two even functions is even.
4. The product of two odd functions is even.
5. The product of an even and an odd function is odd.
6. If f is even,
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx. \tag{3.49}$$
7. If f is odd,
$$\int_{-a}^{a} f(x)\,dx = 0. \tag{3.50}$$



(a) Sine (b) Cosine

Figure 35: Functions Represented by Sine and Cosine Series for $f(x) = x$.

Thus, if $f(x)$ is an even function, the Fourier cosine series represents f not only in $(0, L)$ but also in $(-L, L)$. Similarly, if $f(x)$ is odd, the Fourier sine series represents f not only in $(0, L)$ but also in $(-L, L)$. For example, consider the function $f(x) = x$ in $(0, L)$. Since $\sin x$ is odd and periodic, the sine series for $f(x)$ in $(0, L)$ represents the odd function shown in Fig. 35a. Similarly, since $\cos x$ is even and periodic, the cosine series for $f(x)$ in $(0, L)$ represents the even function shown in Fig. 35b. Both series represent the function in $(0, L)$.

Most functions are neither even nor odd. However, any function $f(x)$ can be represented as the unique sum of even and odd functions

$$f(x) = f_e(x) + f_o(x), \tag{3.51}$$

where $f_e(x)$ is the even part of f given by

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)], \tag{3.52}$$

and $f_o(x)$ is the odd part of f given by

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)]. \tag{3.53}$$

The even part of f has the cosine series representation

$$f_e(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \tag{3.54}$$

where

$$A_0 = \frac{1}{L}\int_0^L f_e(x)\,dx = \frac{1}{2L}\int_{-L}^L f_e(x)\,dx, \tag{3.55}$$

$$A_n = \frac{2}{L}\int_0^L f_e(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^L f_e(x)\cos\frac{n\pi x}{L}\,dx. \tag{3.56}$$

Similarly, the odd part of f has the sine series representation

$$f_o(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}, \tag{3.57}$$

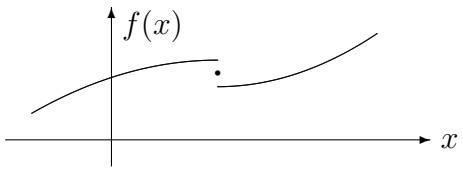


Figure 36: A Function with a Jump Discontinuity.

where

$$B_n = \frac{2}{L}\int_0^L f_o(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^L f_o(x)\sin\frac{n\pi x}{L}\,dx. \tag{3.58}$$

Thus, the complete Fourier series representation of the general function $f(x)$ in the interval $(-L, L)$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{L}, \tag{3.59}$$

where

$$A_0 = \frac{1}{2L}\int_{-L}^L f(x)\,dx, \tag{3.60}$$

$$A_n = \frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,dx, \tag{3.61}$$

$$B_n = \frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{3.62}$$

Note that, in the above three equations for the coefficients, the original function $f(x)$ appears rather than f_e and f_o.

The Fourier series derived above has the following properties:

1. A_0 is the average value of $f(x)$ over the interval $(-L, L)$. If f were an electrical signal, with x representing time, A_0 would thus be the D.C. term.

2. For even functions, $B_n = 0$, and only the cosine series remains.

3. For odd functions, $A_n = 0$, and only the sine series remains.

4. We state without proof that, at points x where $f(x)$ is continuous and differentiable, the series converges to f. At a finite (jump) discontinuity, the series converges to the *average* of the two values of f approached from the left and the right (Fig. 36). This property of Fourier series will be illustrated in §3.6.

3.6 Harmonic Analysis of Pulses

Consider the even function (Fig. 37)

$$f(x) = \begin{cases} A/(2\alpha), & 0 \le |x| < \alpha, \\ 0, & \alpha < |x| \le 1, \end{cases} \tag{3.63}$$

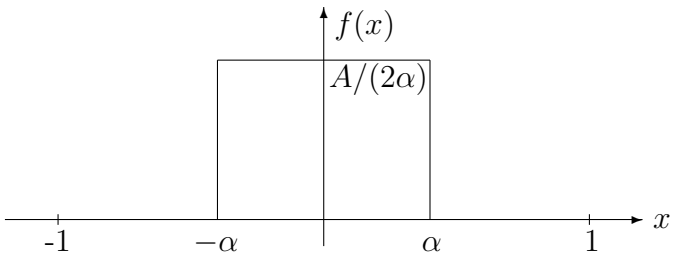


Figure 37: Rectangular Pulse.

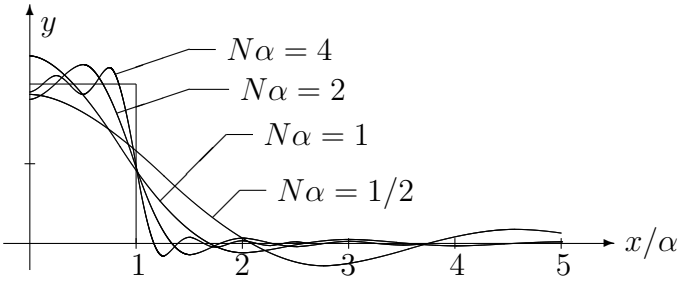


Figure 38: Cosine Series Representations of Rectangular Pulse.

where $0 < \alpha \leq 1$. We consider this function to be a pulse, a short-duration signal. The independent variable x could represent time in an electrical problem or time or distance in a mechanical problem. For example, in linear structural mechanics involving the non-axisymmetric loading of axisymmetric structures, it is common to solve such problems by decomposing both the load and the solution into a Fourier series in the azimuthal coordinate (the angle). In Fig. 37, α is the duration of the pulse as a fraction of the period, and $A/(2\alpha)$ is the height of the pulse, with A the area.

Since $f(x)$ is even, we approximate it with N terms of a Fourier cosine series

$$f(x) \approx A_0 + \sum_{n=1}^{N} A_n \cos n\pi x, \tag{3.64}$$

where

$$A_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = A/2, \tag{3.65}$$

$$A_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \frac{A}{\pi n\alpha}\sin \pi n\alpha. \tag{3.66}$$

The rate at which the series, Eq. 3.64, converges to $f(x)$ is illustrated in Fig. 38, where the abscissa is x/α, and the curves are labeled with $N\alpha$ rather than N to make the figure independent of α and to emphasize the relationship between the number of terms needed and the duration α of the pulse. For example, the curve labeled $N\alpha = 4$ would correspond to 40 terms if $\alpha = 0.1$ and 80 terms if $\alpha = 0.05$. Thus, the number of terms required for a given accuracy varies inversely with the duration of the pulse.

Similar information is obtained from the *frequency spectrum* of the pulse, which plots the magnitude of the Fourier coefficients A_n as a function of the index n (Fig. 39) [26].

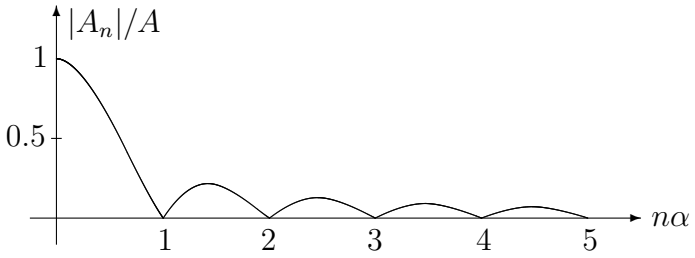


Figure 39: Spectrum of Rectangular Pulse.

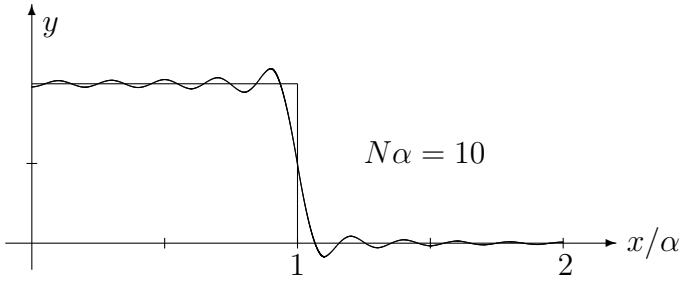


Figure 40: The Gibbs Phenomenom.

(For $n = 0$, $2A_0/A$ is plotted for consistency.) [The word "frequency" comes from electrical engineering, where x represents the time t, and the terms of the Fourier series are harmonics (integer multiples) of the fundamental frequency.] Since the curve in Fig. 39 is displayed with the abscissa $n\alpha$, the curve is applicable to rectangular pulses of various durations. This figure indicates that, for a rectangular pulse, the Fourier coefficients decay slowly, so that a large number of terms may be needed for accurate representation of the pulse. Fig. 39 also indicates that the pulse spectrum has a width which varies inversely with the duration of the pulse, a property common to pulses of all shapes.

Notice also in Fig. 38 that, as the number of terms included in the Fourier series increases, the overshoot which occurs at the discontinuity does not diminish. This property of Fourier series is referred to as the Gibbs phenomenom, which is further illustrated for the same problem in Fig. 40 with a larger number of terms ($N\alpha = 10$) and an enlarged horizontal scale. The amount of the overshoot is approximately 9.5%. This figure also illustrates the convergence of the series representation of the function to the average value of the function at a discontinuity.

Pulse shapes without discontinuities (e.g., triangular, trapezoidal, and cosine pulses) have spectra with faster decay rates. The triangular pulse (Fig. 41), which is continuous but not smooth, is

$$f(x) = \begin{cases} A(\alpha - |x|)/\alpha^2, & 0 \le |x| \le \alpha, \\ 0, & \alpha \le |x| \le 1, \end{cases} \tag{3.67}$$

from which it follows that the Fourier cosine coefficients are

$$A_0 = A/2, \tag{3.68}$$

$$A_n = \frac{4A}{(\pi n\alpha)^2} \sin^2 \frac{\pi n\alpha}{2}. \tag{3.69}$$

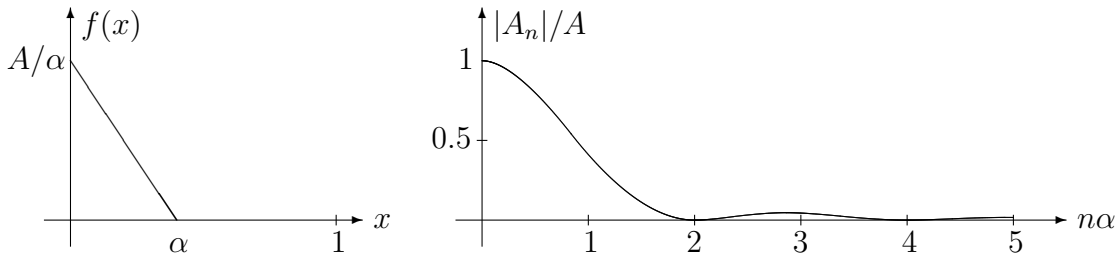


Figure 41: Triangular Pulse and Its Spectrum.

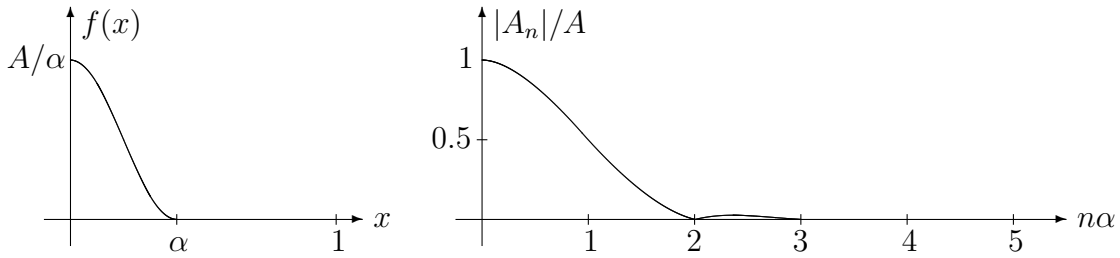


Figure 42: Cosine Pulse and Its Spectrum.

The cosine pulse (Fig. 42), which is smooth, is

$$f(x) = \begin{cases} A[1+\cos(\pi x/\alpha)]/(2\alpha), & 0 \le |x| \le \alpha, \\ 0, & \alpha \le |x| \le 1, \end{cases} \tag{3.70}$$

from which it follows that the Fourier cosine coefficients are

$$A_0 = A/2, \tag{3.71}$$

$$A_n = \frac{A \sin \pi n\alpha}{\pi n\alpha(1-n^2\alpha^2)}, \quad n \neq 1/\alpha, \tag{3.72}$$

$$A_n = A/2, \quad n = 1/\alpha. \tag{3.73}$$

3.7 The Generalized Fourier Series

Consider two functions $f(x)$ and $g(x)$ defined in the interval $[a, b]$ (i.e., $a \le x \le b$). We define the *scalar product* (or *inner product*) of f and g as

$$(f, g) = \int_a^b f(x)g(x)\,dx. \tag{3.74}$$

Note the analogy between this definition for functions and the dot product for vectors.

We define the *norm* N_f of a function f as

$$N_f = (f, f)^{\frac{1}{2}} = \sqrt{\int_a^b [f(x)]^2\,dx}. \tag{3.75}$$

Note that a new function $\bar{f}(x)$ defined as

$$\bar{f}(x) = \frac{f(x)}{N_f} \tag{3.76}$$

has unit norm, i.e., $(\bar{f}, \bar{f}) = 1$. The function $\bar{f}$ is said to be *normalized* over $[a, b]$. Thus, this definition of the norm of a function is analogous to the length of a vector.

We define two functions f and g to be *orthogonal* over $[a, b]$ if $(f, g) = 0$. Orthogonality of functions is analogous to the orthogonality of vectors. We define the set of functions $\phi_i(x),\ i = 1, 2, \cdots,$ to be an *orthonormal* set over $[a, b]$ if $(\phi_i, \phi_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \tag{3.77}$$

For example, Eqs. 3.17 and 3.46 imply that the set of functions

$$\left\{ \frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}} \cos x,\ \frac{1}{\sqrt{\pi}} \sin x,\ \frac{1}{\sqrt{\pi}} \cos 2x,\ \frac{1}{\sqrt{\pi}} \sin 2x,\ \ldots \right\} \tag{3.78}$$

is an orthonormal set over $[0, 2\pi]$.

We now consider the generalization of the trigonometric series, Eq. 3.59. In the series representation of the function $f(x)$,

$$f(x) \approx \sum_{i=1}^{N} c_i \phi_i(x), \tag{3.79}$$

we wish to evaluate the coefficients c_i so as to minimize the mean square error

$$M_N = \int_a^b \left[f(x) - \sum_{i=1}^{N} c_i \phi_i(x) \right]^2 dx, \tag{3.80}$$

where $(\phi_i, \phi_j) = \delta_{ij}$.

To minimize M_N, we set, for each j,

$$0 = \frac{\partial M_N}{\partial c_j} = \int_a^b 2 \left[f(x) - \sum_{i=1}^{N} c_i \phi_i(x) \right] [-\phi_j(x)]\, dx, \tag{3.81}$$

or

$$\int_a^b f(x) \phi_j(x)\, dx = \sum_{i=1}^{N} c_i \int_a^b \phi_i(x) \phi_j(x)\, dx = \sum_{i=1}^{N} c_i \delta_{ij} = c_j. \tag{3.82}$$

That is,

$$c_i = (f, \phi_i). \tag{3.83}$$

Alternatively, had we started with the series, Eq. 3.79, we would have obtained the same result by taking the inner product of each side with ϕ_j:

$$(f, \phi_j) = \sum_{i=1}^{N} c_i (\phi_i, \phi_j) = \sum_{i=1}^{N} c_i \delta_{ij} = c_j. \tag{3.84}$$

The expansion coefficients c_i are called the *Fourier coefficients*, and the series is called the *generalized Fourier series*. The generalized Fourier series, like the trigonometric series, has the property that the series representation of a function minimizes the mean square error between the series and the function. We therefore see that a Fourier series is a series expansion in orthogonal functions that minimizes the mean square error between the series representation and the given function.

The formula for the generalized Fourier coefficient c_i in Eq. 3.83 can be interpreted as the projection of the function $f(x)$ on the ith basis function ϕ_i. This interpretation is analogous to the expansion of a vector in terms of the unit basis vectors in the coordinate directions. For example, in three-dimensions, the vector $\mathbf{v}$ is, in Cartesian coordinates,

$$\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y + v_z \mathbf{e}_z, \tag{3.85}$$

where the components v_x, v_y, and v_z are the projections of $\mathbf{v}$ onto the coordinate basis vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$. That is,

$$v_x = \mathbf{v} \cdot \mathbf{e}_x, \;\; v_y = \mathbf{v} \cdot \mathbf{e}_y, \;\; v_z = \mathbf{v} \cdot \mathbf{e}_z. \tag{3.86}$$

These components are analogous to the Fourier coefficients c_i.

From Eq. 3.80, we can deduce some additional properties of the Fourier series. From that equation, the mean square error after summing N terms of the series is

$$\begin{aligned} M_N &= (f,f) - 2\sum_{i=1}^{N} c_i (f,\phi_i) + \int_a^b \left(\sum_{i=1}^{N} c_i \phi_i \right) \left(\sum_{j=1}^{N} c_j \phi_j \right) dx \\ &= (f,f) - 2\sum_{i=1}^{N} c_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} c_i c_j (\phi_i, \phi_j) \\ &= (f,f) - 2\sum_{i=1}^{N} c_i^2 + \sum_{i=1}^{N} c_i^2 \\ &= (f,f) - \sum_{i=1}^{N} c_i^2 \end{aligned} \tag{3.87}$$

Since, by definition, $M_N \geq 0$, this last equation implies

$$\sum_{i=1}^{N} c_i^2 \leq (f,f), \tag{3.88}$$

which is referred to as Bessel's Inequality. Since the right-hand side of this inequality (the square of the norm of f) is fixed for a given function, and each term on the left-hand side is non-negative, a necessary condition for a Fourier series to converge is that $c_i \to 0$ as $i \to \infty$.

From Eq. 3.87, the error after summing N terms of the series is

$$M_N = (f,f) - \sum_{i=1}^{N} c_i^2, \tag{3.89}$$

so that the error after $N+1$ terms is

$$M_{N+1} = (f,f) - \sum_{i=1}^{N+1} c_i^2 = (f,f) - \sum_{i=1}^{N} c_i^2 - c_{N+1}^2 = M_N - c_{N+1}^2. \tag{3.90}$$

That is, adding a new nonzero term in the series decreases the error. Thus, in general, higher order approximations are better than lower order approximations.

The set of basis functions $\phi_i(x)$ is defined as a *complete* set if

$$\lim_{N\to\infty} M_N = \lim_{N\to\infty} \int_a^b \left[f(x) - \sum_{i=1}^{N} c_i \phi_i(x) \right]^2 dx = 0 \tag{3.91}$$

for arbitrary $f(x)$. If this equation holds, the Fourier series is said to *converge in the mean*. Having a complete set means that the set of basis functions $\phi_i(x)$ is not missing any members. Thus, we could alternatively define the set $\phi_i(x)$ as complete if there is no function with positive norm which is orthogonal to each basis function $\phi_i(x)$ in the set. It turns out that the set of functions shown in Eq. 3.78 forms a complete set in $[0, 2\pi]$.

Note that convergence in the mean does not imply convergence pointwise. That is, it may not be true that

$$\lim_{N\to\infty} \left| f(x) - \sum_{i=1}^{N} c_i \phi_i(x) \right| = 0 \tag{3.92}$$

for all x. For example, the series representation of a discontinuous function $f(x)$ will not have pointwise convergence at the discontinuity (Fig. 36).

The vector analog to completeness for sets of functions can be illustrated by observing that, to express an arbitrary three-dimensional vector in terms of a sum of coordinate basis vectors, three basis vectors are needed. A basis consisting of only two unit vectors is incomplete, since it in incapable of representing arbitrary vectors in three dimensions.

4 Problems in Cartesian Coordinates

4.1 One-Dimensional Problems

The heat conduction problem solved in §3.1 was homogeneous and had zero boundary data. The only forcing function (a nonhomogeneous term which forces the solution to be nonzero) was the nonzero initial condition. Here we consider the same problem, one-dimensional heat conduction, with both a nonzero boundary condition and a nonzero initial condition (Fig. 43). The mathematical statement of this problem is

$$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{K}\dfrac{\partial T}{\partial t}, \ 0 < x < L, \\ T(0,t) = 0, \ T(L,t) = T_0, \ T(x,0) = f(x), \end{cases} \tag{4.1}$$

where L is the length of the bar, T_0 is a constant, and $f(x)$ is the initial distribution of temperature prescribed over the length of the bar. The problem is to find $T(x,t)$ for $t \geq 0$. Given the discussion in §2.7, we note that this is a well-posed problem.

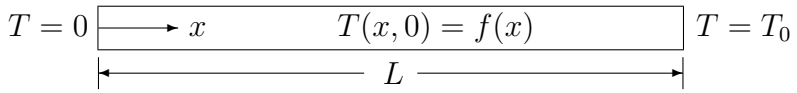


Figure 43: 1-D Heat Conduction With Nonzero B.C. and I.C.

The problem solved in §3.1 was similar except that $T_0 = 0$ in that problem. An attempt to solve Eq. 4.1 directly using separation of variables would lead to a result similar to Eq. 3.13:

$$T(x,t) = A \sin \alpha x \; e^{-K\alpha^2 t}, \tag{4.2}$$

where A and α are unknown constants. This function satisfies the partial differential equation and the boundary condition at $x = 0$. To satisfy the nonhomogeneous boundary condition at $x = L$, we require

$$T(L,t) = T_0 = A \sin \alpha L \; e^{-K\alpha^2 t}, \tag{4.3}$$

an equation which cannot be satisfied by picking the constants A and α.

It has been found useful, for problems with multiple forcing functions, to decompose the solution into the sum of two or more solutions, one for each forcing function. For Eq. 4.1, we note that one forcing function (T_0) is time-independent, so we might expect to be able to satisfy this boundary condition with a steady-state (time-independent) function. Since there is also a transient part of the problem, at least one part of the problem must allow a time-dependent solution. We therefore attempt a solution which is the sum of transient and steady-state parts:

$$T(x,t) = T_1(x,t) + T_2(x). \tag{4.4}$$

The substitution of Eq. 4.4 into Eq. 4.1 yields

$$\begin{cases} \dfrac{\partial^2 T_1}{\partial x^2} + T_2'' = \dfrac{1}{K}\dfrac{\partial T_1}{\partial t}, \\ T_1(0,t) + T_2(0) = 0, \\ T_1(L,t) + T_2(L) = T_0, \\ T_1(x,0) + T_2(x) = f(x). \end{cases} \tag{4.5}$$

This system can then be separated into two systems:

$$T_2''(x) = 0, \; T_2(0) = 0, \; T_2(L) = T_0, \tag{4.6}$$

and

$$\begin{cases} \dfrac{\partial^2 T_1}{\partial x^2} = \dfrac{1}{K}\dfrac{\partial T_1}{\partial t}, \\ T_1(0,t) = 0, \; T_1(L,t) = 0, \; T_1(x,0) = f(x) - T_2(x), \end{cases} \tag{4.7}$$

where T_1 and T_2 are each solutions of systems with one forcing function. Notice that the T_1 solution depends on finding the steady-state solution $T_2(x)$.

Eq. 4.6 can be integrated directly to obtain

$$T_2(x) = T_0 x/L. \tag{4.8}$$

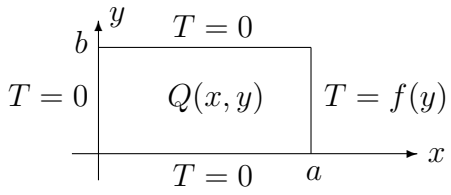


Figure 44: Rectangular Domain With Dirichlet B.C.

The T_1 solution is already known, since the solution of Eq. 4.7 is the same as that of Eq. 3.1 with $f(x)$ replaced by $f(x) - T_2(x)$. The solution of Eq. 4.1 is therefore

$$T(x,t) = \frac{T_0 x}{L} + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-K(n\pi/L)^2 t}, \tag{4.9}$$

where

$$A_n = \frac{2}{L}\int_0^L \left[f(x) - \frac{T_0 x}{L}\right] \sin\frac{n\pi x}{L}\, dx. \tag{4.10}$$

From this problem, we see that, in problems with multiple forcing functions, it is useful to decompose the solution into multiple parts, one for each nonzero function which "forces" the solution. Each component solution satisfies a system with only one forcing function. Also, if some of the forcing functions are time-independent, it is useful, as part of this decomposition, to separate the problem into steady-state and transient parts. The transient part of the problem will therefore be forced by the difference between the original initial condition and the steady-state solution. Since the steady-state solution for a problem is the long-time solution, the transient part of the solution would be expected to decay for large time.

4.2 Two-Dimensional Problems

Consider the following two-dimensional problem, where the goal is to find $T(x,y,t)$ in the interior of the rectangle $0 \le x \le a$, $0 \le y \le b$ for $t \ge 0$:

$$\begin{cases} \nabla^2 T + \dfrac{Q}{k} = \dfrac{1}{K}\dfrac{\partial T}{\partial t}, \\ T(x,0,t) = 0,\ T(x,b,t) = 0,\ T(0,y,t) = 0,\ T(a,y,t) = f(y), \\ T(x,y,0) = T_0(x,y), \end{cases} \tag{4.11}$$

where $Q = Q(x,y)$ (Fig. 44). In two dimensions in Cartesian coordinates,

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}. \tag{4.12}$$

The increase in complexity of this problem compared with the problem of the preceding section is due to the increase in spatial dimensions and the increase in the number of forcing functions.

The problem in Eq. 4.11 has three forcing functions: the internal source $Q(x, y)$, the nonzero boundary condition $f(y)$, and the initial condition $T_0(x, y)$. We therefore attempt a solution as the sum of three functions, one for each forcing function:

$$T(x, y, t) = u(x, y, t) + v(x, y) + w(x, y). \tag{4.13}$$

We intend for u to be the transient part of the solution, v to have the source Q as its forcing function, and w to have the nonzero boundary condition f as its forcing function. The substitution of Eq. 4.13 into Eq. 4.11 yields

$$\begin{cases} \nabla^2 u + \nabla^2 v + \nabla^2 w + \dfrac{Q}{k} = \dfrac{1}{K}\dfrac{\partial u}{\partial t}, \\ u(x, 0, t) + v(x, 0) + w(x, 0) = 0, \\ u(x, b, t) + v(x, b) + w(x, b) = 0, \\ u(0, y, t) + v(0, y) + w(0, y) = 0, \\ u(a, y, t) + v(a, y) + w(a, y) = f(y), \\ u(x, y, 0) + v(x, y) + w(x, y) = T_0(x, y). \end{cases} \tag{4.14}$$

The original problem can thus be decomposed into the three problems

$$\text{I.} \begin{cases} \nabla^2 u = \dfrac{1}{K}\dfrac{\partial u}{\partial t}, \\ u(x, 0, t) = 0,\ u(x, b, t) = 0,\ u(0, y, t) = 0,\ u(a, y, t) = 0, \\ u(x, y, 0) = T_0(x, y) - v(x, y) - w(x, y), \end{cases} \tag{4.15}$$

$$\text{II.} \begin{cases} \nabla^2 v = -Q/k, \\ v(x, 0) = v(x, b) = v(0, y) = v(a, y) = 0, \end{cases} \tag{4.16}$$

$$\text{III.} \begin{cases} \nabla^2 w = 0, \\ w(x, 0) = w(x, b) = w(0, y) = 0,\ w(a, y) = f(y). \end{cases} \tag{4.17}$$

Notice that, if the initial condition $T_0(x, y)$ were zero, there would still be a transient part of the problem, since the initial condition for the transient part of the problem, Eq. 4.15c, is the difference between the original initial condition and the steady-state solution. We now consider each of these three problems in turn.

4.2.1 The Transient Problem

Since the PDE in Eq. 4.15 is homogeneous, we will look for a solution using separation of variables. Although we could do the separation in one step by substituting $u(x, y, t) = X(x)Y(y)\mathcal{T}(t)$, we will instead first separate between space and time:

$$u(x, y, t) = w(x, y)\mathcal{T}(t), \tag{4.18}$$

which, when substituted into Eq. 4.15a yields

$$(\nabla^2 w)\mathcal{T} = \frac{1}{K} w \mathcal{T}' \tag{4.19}$$

or

$$\frac{\nabla^2 w}{w} = \frac{1}{K}\frac{\mathcal{T}'}{\mathcal{T}}. \tag{4.20}$$

For a function of x and y to be identically equal to a function of t for all x, y, and t, both sides of this equation must equal a constant. On the basis of our one-dimensional experience in §3.1, in which the disturbances decayed in time, we would anticipate a negative separation constant:

$$\frac{\nabla^2 w}{w} = \frac{1}{K}\frac{\mathcal{T}'}{\mathcal{T}} = -\alpha^2. \tag{4.21}$$

Thus, the separated equations are

$$\mathcal{T}' + K\alpha^2\mathcal{T} = 0, \tag{4.22}$$

$$\nabla^2 w + \alpha^2 w = 0. \tag{4.23}$$

The space equation, Eq. 4.23, is the Helmholtz equation, Eq. 2.18. Although the Helmholtz equation arose here as the spatial part of a transient heat conduction problem, it arises most commonly in steady-state (time-harmonic) wave problems (e.g., acoustics). Notice that this equation arises here regardless of the coordinate system used. The solution w can be further separated into the product

$$w(x, y) = X(x)Y(y), \tag{4.24}$$

which, when substituted into Eq. 4.23 yields

$$X''Y + XY'' + \alpha^2 XY = 0 \tag{4.25}$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y} - \alpha^2 = \beta^2, \tag{4.26}$$

where β^2 is the second separation constant for this problem. The sign associated with this constant remains to be verified. Thus,

$$\begin{cases} X'' - \beta^2 X = 0 \\ Y'' + (\alpha^2 + \beta^2)Y = 0. \end{cases} \tag{4.27}$$

The boundary conditions, Eq. 4.15b, imply

$$X(0) = X(a) = 0,\ Y(0) = Y(b) = 0. \tag{4.28}$$

The general solution of Eq. 4.27a is

$$X(x) = A\sinh\beta x + B\cosh\beta x, \tag{4.29}$$

where the boundary conditions, Eq. 4.28a, imply $A = B = 0$, $X(x) = 0$, and $u(x, y, t) = 0$. Since $u = 0$ does not satisfy the initial data for Eq. 4.15, we conclude that the sign of the second separation constant, β^2, was chosen incorrectly. If we thus replace β^2 with $-\beta^2$, the separated ordinary differential equations are

$$\begin{cases} X'' + \beta^2 X = 0 \\ Y'' + (\alpha^2 - \beta^2)Y = 0. \end{cases} \tag{4.30}$$

The general solution of the new x equation is

$$X(x) = A \sin \beta x + B \cos \beta x, \tag{4.31}$$

where the boundary data, Eq. 4.28a, imply $B = 0$ and $\sin \beta a = 0$. Thus,

$$\beta a = m\pi, \ \beta = \frac{m\pi}{a}, \ m = 1, 2, 3, \ldots \tag{4.32}$$

and

$$X(x) = \sin \frac{m\pi x}{a}. \tag{4.33}$$

We now address the y equation, Eq. 4.30b. For convenience, we define

$$\gamma^2 = \alpha^2 - \beta^2. \tag{4.34}$$

The general solution of Eq. 4.30b is then

$$Y(y) = C \sin \gamma y + D \cos \gamma y, \tag{4.35}$$

where the boundary data, Eq. 4.28b, imply $D = 0$ and $\sin \gamma b = 0$. Thus,

$$\gamma = \frac{n\pi}{b}, \ n = 1, 2, 3, \ldots \tag{4.36}$$

and

$$Y(y) = \sin \frac{n\pi y}{b} \tag{4.37}$$

The solution of the time equation, Eq. 4.22, is

$$\mathcal{T}(t) = e^{-K\alpha^2 t}, \tag{4.38}$$

where

$$\alpha^2 = \beta^2 + \gamma^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{4.39}$$

The three solutions $X(x)$, $Y(y)$, and $\mathcal{T}(t)$ can now be combined to yield

$$u(x, y, t) = X(x)Y(y)\mathcal{T}(t) = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-K[(m\pi/a)^2 + (n\pi/b)^2]t}, \tag{4.40}$$

which satisfies the partial differential equation and the boundary conditions, but not the initial condition

$$u(x, y, 0) = T_0(x, y) - v(x, y) - w(x, y) = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{4.41}$$

However, since the function $u(x, y, t)$ given above satisfies the partial differential equation and boundary conditions for all positive integer values of m and n, a linear combination of all such functions also satisfies the partial differential equation and boundary conditions:

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-K[(m\pi/a)^2 + (n\pi/b)^2]t}, \tag{4.42}$$

where the coefficients A_{mn} may be different for each term in the series.

The initial condition thus requires that

$$u(x,y,0) = T_0(x,y) - v(x,y) - w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}. \tag{4.43}$$

The orthogonality relations, Eq. 3.17, can be used to evaluate the coefficients A_{mn}. We first multiply the above equation by $\sin(\bar{m}\pi x/a)$, and integrate from 0 to a, to obtain

$$\begin{aligned}\int_0^a (T_0 - v - w) \sin\frac{\bar{m}\pi x}{a}\, dx &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi y}{b} \int_0^a \sin\frac{m\pi x}{a} \sin\frac{\bar{m}\pi x}{a}\, dx \\ &= \frac{a}{2}\sum_{n=1}^{\infty} A_{\bar{m}n} \sin\frac{n\pi y}{b}. \end{aligned}\tag{4.44}$$

We now multiply this result by $\sin(\bar{n}\pi y/b)$, and integrate from 0 to b, to obtain

$$\int_0^b \int_0^a (T_0 - v - w) \sin\frac{\bar{m}\pi x}{a} \sin\frac{\bar{n}\pi y}{b}\, dx\, dy = \frac{a}{2} A_{\bar{m}\bar{n}} \frac{b}{2}, \tag{4.45}$$

or

$$A_{mn} = \frac{4}{ab}\int_0^b \int_0^a (T_0 - v - w) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy. \tag{4.46}$$

Thus, the solution to Problem I (the u equation) is Eq. 4.42 with the coefficients A_{mn} given by Eq. 4.46.

4.2.2 The Steady-State Problem with Nonzero Source

The forcing function for Problem II (the v equation) appears in the partial differential equation, Eq. 4.16a, a nonhomogeneous equation. It is not possible to solve this equation (and other nonhomogeneous partial differential equations) using separation of variables, since nonhomogeneous equations cannot be separated. However, since $Q(x,y)$ is arbitrary, it is likely that we will have to expand both it and the solution v in terms of orthogonal functions. Thus, we seek a general function which is zero on the boundary of the rectangle and can be used in an expansion. Since the product

$$\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

is zero on the boundary and has been used in an expansion in Eq. 4.43, we try

$$v(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}. \tag{4.47}$$

If the coefficients B_{mn} can be determined such that Eq. 4.47 satisfies Eq. 4.16a, we have the (unique) solution of Problem II.

We substitute Eq. 4.47 into Eq. 4.16a to obtain

$$-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=-\frac{Q(x,y)}{k}. \tag{4.48}$$

We now multiply both sides of this equation by $\sin(\bar{m}\pi x/a)\sin(\bar{n}\pi y/b)$, and integrate:

$$B_{\bar{m}\bar{n}}\left[\left(\frac{\bar{m}\pi}{a}\right)^2+\left(\frac{\bar{n}\pi}{b}\right)^2\right]\frac{a}{2}\frac{b}{2}=\int_0^b\int_0^a\frac{Q(x,y)}{k}\sin\frac{\bar{m}\pi x}{a}\sin\frac{\bar{n}\pi y}{b}\,dx\,dy, \tag{4.49}$$

or

$$B_{mn}=\frac{4}{ab\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]}\int_0^b\int_0^a\frac{Q(x,y)}{k}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy. \tag{4.50}$$

That is, the solution of Problem II, Eq. 4.16, is the series, Eq. 4.47, with the coefficients given in Eq. 4.50. Since this is a well-posed problem, the solution is unique.

If we had not solved the transient part of the problem first, we would not have known what functions to use in the expansion of Eq. 4.47. In that case, and in general, we first solve the Helmholtz equation, Eq. 4.23, with zero boundary data to find suitable expansion functions. This expansion is referred to as an eigenfunction expansion, since the expansion functions are eigenfunctions of the Helmholtz equation.

4.2.3 The Steady-State Problem with Nonzero Boundary Data

The forcing function for Problem III (the w equation, Eq. 4.17) appears in the nonzero boundary data. Since the partial differential equation is homogeneous, we attempt a solution using the technique of separation of variables. The substitution of

$$w(x,y)=X(x)Y(y) \tag{4.51}$$

in Eq. 4.17a yields

$$X''Y+XY''=0, \tag{4.52}$$

or

$$\frac{X''}{X}=-\frac{Y''}{Y}=\delta^2, \tag{4.53}$$

where the sign of the separation constant δ^2 was selected to yield trigonometric solutions in the y direction, since we anticipate the need for a Fourier series expansion of $f(y)$.

The separated ordinary differential equations are thus

$$\begin{cases} X''-\delta^2X=0, \\ Y''+\delta^2Y=0. \end{cases} \tag{4.54}$$

The general solution of the y equation is

$$Y(y)=E\sin\delta y+F\cos\delta y, \tag{4.55}$$

where the boundary data, Eq. 4.17b, imply $F = 0$ and $\sin \delta b = 0$. Thus,

$$\delta b = \ell\pi, \ \delta = \frac{\ell\pi}{b}, \ \ell = 1, 2, 3, \ldots \tag{4.56}$$

and

$$Y(y) = \sin \frac{\ell\pi y}{b}. \tag{4.57}$$

The general solution of the x equation is

$$X(x) = G \sinh \delta x + H \cosh \delta x, \tag{4.58}$$

where the boundary condition $X(0) = 0$ implies $H = 0$. Hence

$$X(x) = \sinh \frac{\ell\pi x}{b}, \tag{4.59}$$

and the general solution of Problem III is thus

$$w(x, y) = \sum_{\ell=1}^{\infty} E_\ell \sinh \frac{\ell\pi x}{b} \sin \frac{\ell\pi y}{b}. \tag{4.60}$$

The nonzero boundary condition, Eq. 4.17, requires

$$w(a, y) = f(y) = \sum_{\ell=1}^{\infty} E_\ell \sinh \frac{\ell\pi a}{b} \sin \frac{\ell\pi y}{b}. \tag{4.61}$$

The multiplication of both sides of this equation by $\sin(\bar{\ell}\pi y/b)$ and integrating from 0 to b yields

$$E_\ell = \frac{2}{b \sinh(\ell\pi a/b)} \int_0^b f(y) \sin \frac{\ell\pi y}{b} \, dy, \tag{4.62}$$

thus completing the solution of Problem III and the entire problem, Eq. 4.11.

We note that the homogeneous partial differential equations above were solved using the technique of separation of variables, and the nonhomogeneous partial differential equation was solved using an expansion in orthogonal functions.

5 Sturm-Liouville System

The evaluation of Fourier coefficients in the examples considered so far in this volume has involved orthogonality relationships for the trigonometric functions, Eqs. 3.17 and 3.46. For more general problems in Cartesian and other coordinate systems, a more general treatment of orthogonality is needed. The Sturm-Liouville system provides such generality.

We consider the general, second-order, linear, homogeneous ordinary differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + [q(x) + \lambda r(x)]y(x) = 0 \tag{5.1}$$

in the domain (a, b) with general *homogeneous* boundary conditions of the form

$$\begin{aligned} Ay'(a) + By(a) &= 0, \\ Cy'(b) + Dy(b) &= 0. \end{aligned} \tag{5.2}$$

Values of the constant λ for which nontrivial (nonzero) solutions exist are called *eigenvalues* of the Sturm-Liouville system, and the corresponding nontrivial solutions $y(x)$ satisfying Eqs. 5.1 and 5.2 are called *eigenfunctions*. The trivial solution $y(x) = 0$, which also satisfies the Sturm-Liouville problem, is of no interest.

We now establish the orthogonality relations for the eigenfunctions. Assume we have two solutions (eigenvalue-eigenfunction pairs) of the Sturm-Liouville system. If ϕ_n is an eigenfunction corresponding to the eigenvalue λ_n,

$$\begin{cases} \dfrac{d}{dx}[p\phi_n'] + [q + \lambda_n r]\phi_n = 0, \\ A\phi_n'(a) + B\phi_n(a) = 0, \\ C\phi_n'(b) + D\phi_n(b) = 0. \end{cases} \tag{5.3}$$

Similarly, for the eigenfunction ϕ_m corresponding to λ_m,

$$\begin{cases} \dfrac{d}{dx}[p\phi_m'] + [q + \lambda_m r]\phi_m = 0, \\ A\phi_m'(a) + B\phi_m(a) = 0, \\ C\phi_m'(b) + D\phi_m(b) = 0. \end{cases} \tag{5.4}$$

If we now multiply Eq. 5.3b by $\phi_m(a)$ and Eq. 5.4b by $\phi_n(a)$, and subtract, we obtain

$$\phi_n(a)\phi_m'(a) - \phi_n'(a)\phi_m(a) = 0. \tag{5.5}$$

Similarly, at $x = b$,

$$\phi_n(b)\phi_m'(b) - \phi_n'(b)\phi_m(b) = 0. \tag{5.6}$$

Also, from the differential equations, if we multiply Eq. 5.3a by $\phi_m(x)$ and Eq. 5.4a by $\phi_n(x)$, and subtract, we obtain

$$(\lambda_n - \lambda_m)r(x)\phi_n(x)\phi_m(x) = \phi_n\frac{d}{dx}[p\phi_m'] - \phi_m\frac{d}{dx}[p\phi_n'] = \frac{d}{dx}[p(\phi_n\phi_m' - \phi_m\phi_n')], \tag{5.7}$$

where the last equation can be verified by expanding the right-hand side. We now integrate this last result to obtain

$$(\lambda_n - \lambda_m)\int_a^b r(x)\phi_n(x)\phi_m(x)\,dx = \{p(x)[\phi_n(x)\phi_m'(x) - \phi_m(x)\phi_n'(x)]\}\Big|_a^b = 0, \tag{5.8}$$

where the second equality follows from Eqs. 5.5 and 5.6. Thus, for distinct eigenvalues of the Sturm-Liouville problem ($\lambda_n \neq \lambda_m$),

$$\int_a^b r(x)\phi_n(x)\phi_m(x)\,dx = 0. \tag{5.9}$$

That is, eigenfunctions $\phi_n(x)$ and $\phi_m(x)$ corresponding to distinct eigenvalues are *orthogonal* over (a, b) with respect to the weighting function $r(x)$.

Remarks:

1. From Eq. 5.8, we would also have orthogonality if $p(a) = p(b) = 0$. This property will turn out to be useful in spherical coordinates for problems involving Legendre's Equation or the Associated Legendre's Equation.

2. From Eq. 5.8, we would also have orthogonality if $p(a) = p(b)$ and the boundary conditions in Eq. 5.2 are replaced by the conditions $y(a) = y(b)$ and $y'(a) = y'(b)$. This property will turn out to be useful in cylindrical coordinates for domains having no boundaries in one direction, e.g., circular domains where the azimuthal coordinate extends from 0 to 2π.

3. If $\lambda_n = \lambda_m$, but ϕ_n and ϕ_m are linearly independent, the eigenvalues are said to be *degenerate*, in which case ϕ_n and ϕ_m need not be orthogonal. That is, in the degenerate case, there is no guarantee of orthogonality.

4. It can be shown that, if $r(x) > 0$, all the eigenvalues are real.

5.1 Examples of Sturm-Liouville Problems

Separation of variables solutions in Cartesian coordinates for the heat equation resulted in ordinary differential equations of the form

$$y'' + \lambda^2 y = 0, \tag{5.10}$$

with solutions $\sin \lambda x$ and $\cos \lambda x$. This equation is in Sturm-Liouville form with $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. If $y(x)$ satisfies homogeneous boundary conditions of Sturm-Liouville type, Eq. 5.2, the solutions are orthogonal with respect to $r(x) = 1$.

Some separation of variables solutions in spherical coordinates result in Legendre's Equation,

$$(1 - x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \quad (-1 \le x \le 1), \tag{5.11}$$

or, in Sturm-Liouville form,

$$\frac{d}{dx}[(1 - x^2)P_n'] + n(n+1)P_n = 0 \quad (-1 \le x \le 1), \tag{5.12}$$

which implies that $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$, and $\lambda_n = n(n+1)$. Solutions of Legendre's Equation are denoted $P_n(x)$, which is regular over $(-1, 1)$, and $Q_n(x)$, which is singular at $x = \pm 1$. Since $p(\pm 1) = 0$, orthogonality of the Legendre functions $P_n(x)$ follows from Remark 1 above. It will turn out than $P_n(x)$ is a polynomial of degree n.

Some separation of variables solutions in cylindrical coordinates result in Bessel's Equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2 x^2 - \nu^2)y = 0 \quad (\nu \text{ specified}), \tag{5.13}$$

or, in Sturm-Liouville form,

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(\lambda^2 x - \frac{\nu^2}{x}\right)y = 0, \tag{5.14}$$

which implies $p(x) = x$, $q(x) = -\nu^2/x$, $r(x) = x$, and the eigenvalue is λ^2. For $\nu = n$ an integer, the case of interest in this volume, the solutions are denoted $J_n(\lambda x)$ and $Y_n(\lambda x)$, which are called Bessel's Functions of the First and Second Kind, respectively. If $y(x)$ satisfies homogeneous boundary conditions of Sturm-Liouville type, Eq. 5.2, the solutions are orthogonal with respect to the weighting $r(x) = x$.

Some separation of variables solutions in spherical coordinates result in the Associated Legendre's Equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0 \quad (-1 \le x \le 1), \tag{5.15}$$

where $n = 0, 1, 2, \ldots$, and $m = 0, 1, 2, \ldots$ (m fixed), or, in Sturm-Liouville form,

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0, \tag{5.16}$$

which implies that $p(x) = 1 - x^2$, $q(x) = -m^2/(1-x^2)$, $r(x) = 1$, and $\lambda_n = n(n+1)$. It can be shown that solutions of the Associated Legendre's Equation are

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m P_n(x)}{dx^m}, \tag{5.17}$$

which is regular over $(-1, 1)$, and $Q_n^m(x)$, which has singularities. Since the eigenfunctions $P_n^m(x)$ are solutions of the Associated Legendre's Equation for different values of n for a given value of m, the orthogonality relationship for these functions is

$$\int_{-1}^{1} P_\alpha^m(x) P_\beta^m(x)\, dx = 0 \tag{5.18}$$

for fixed m and $\alpha \neq \beta$. Thus, to expand a function $f(x)$ in terms of the $P_n^m(x)$, we have

$$f(x) = \sum_{\alpha=0}^{\infty} c_\alpha P_\alpha^m(x) \quad (-1 \le x \le 1). \tag{5.19}$$

To compute the coefficients c_α, we multiply both sides of this equation by $P_\beta^m(x)$, and integrate from -1 to 1 to obtain

$$\int_{-1}^{1} f(x) P_\beta^m(x)\, dx = \sum_{\alpha=0}^{\infty} c_\alpha \int_{-1}^{1} P_\alpha^m(x) P_\beta^m(x)\, dx = c_\beta \int_{-1}^{1} [P_\beta^m(x)]^2\, dx. \tag{5.20}$$

Hence,

$$c_\alpha = \frac{\int_{-1}^{1} f(x) P_\alpha^m(x)\, dx}{\int_{-1}^{1} [P_\alpha^m(x)]^2\, dx}. \tag{5.21}$$

According to the generalized Fourier series theory discussed in §3.7, these coefficients yield a series representation of $f(x)$ which minimizes the mean square error between the function and the series.

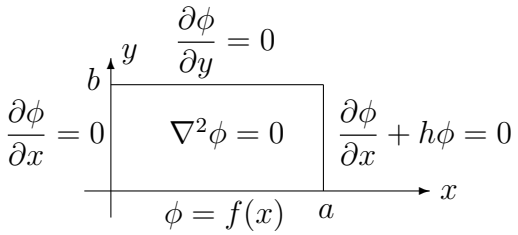


Figure 45: Rectangular Domain With Gradient B.C.

5.2 Boundary Value Problem Example

Consider the following two-dimensional boundary value problem, where the goal is to find $\phi(x, y)$ in the interior of the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$:

$$\begin{aligned} &\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \\ &\frac{\partial \phi}{\partial x} = 0 \text{ for } x = 0,\ \phi = f(x) \text{ for } y = 0,\ \frac{\partial \phi}{\partial y} = 0 \text{ for } y = b,\ \frac{\partial \phi}{\partial x} + h\phi = 0 \text{ for } x = a, \end{aligned} \tag{5.22}$$

where $h > 0$ (Fig. 45). This problem has one forcing function, the boundary condition specified at $y = 0$. The boundary condition specified at $x = a$ is homogeneous and hence not forcing. This boundary condition is analogous to a convection boundary condition, since the heat flux is proportional to temperature.

The solution of this problem is similar to that of the problem solved in §4.2.3. We attempt a solution using the technique of separation of variables. The substitution of

$$\phi(x, y) = X(x)Y(y) \tag{5.23}$$

in Eq. 5.22a yields

$$X''Y + XY'' = 0, \tag{5.24}$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\alpha^2, \tag{5.25}$$

where the sign of the separation constant α^2 was selected to yield trigonometric solutions in the x direction, since we anticipate the need for a Fourier series expansion of $f(x)$.

The separated ordinary differential equations are thus

$$\begin{cases} X'' + \alpha^2 X = 0, \\ Y'' - \alpha^2 Y = 0. \end{cases} \tag{5.26}$$

The general solution of the x equation is

$$X(x) = A \sin \alpha x + B \cos \alpha x, \tag{5.27}$$

whose derivative (needed in boundary conditions) is

$$X'(x) = A\alpha \cos \alpha x - B\alpha \sin \alpha x. \tag{5.28}$$

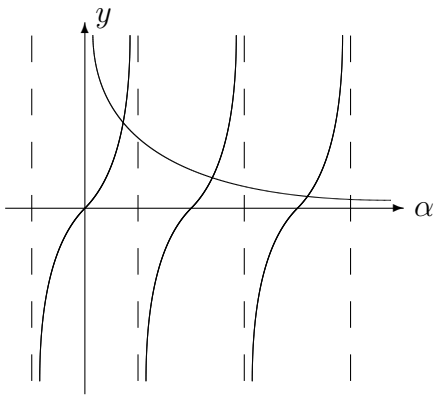


Figure 46: Graphical Solution of $\tan \alpha a = h/\alpha$.

The boundary condition $X'(0) = 0$ implies

$$A\alpha = 0, \tag{5.29}$$

so that either $A = 0$ or $\alpha = 0$.

Since

$$\frac{\partial \phi}{\partial x} + h\phi = X'(x)Y(y) + hX(x)Y(y) = [X'(x) + hX(x)]Y(y), \tag{5.30}$$

and assuming the first case ($A = 0$), the boundary condition at $x = a$ implies

$$0 = X'(a) + hX(a) = -B\alpha \sin \alpha a + hB \cos \alpha a, \tag{5.31}$$

or

$$\tan \alpha a = \frac{h}{\alpha}, \tag{5.32}$$

whose solutions are α_n. That is,

$$\tan \alpha_n a = \frac{h}{\alpha_n}, \quad n = 1, 2, 3, \dots \ , \tag{5.33}$$

and

$$X(x) = \cos \alpha_n x. \tag{5.34}$$

Note that n in this case is simply a counter. The solutions of Eq. 5.33 are shown graphically as the points of intersection of the tangent curves and the hyperbola in Fig. 46.

To address the second case implied by Eq. 5.29, $\alpha = 0$, we must solve the ordinary differential equation, Eq. 5.26a, again, since the two solutions in Eq. 5.27 are no longer independent. The differential equation $X''(x) = 0$ has as its general solution

$$X(x) = cx + d. \tag{5.35}$$

However, since the boundary conditions at $x = 0$ and $x = a$ imply $c = d = 0$, the case $\alpha = 0$ is of no further interest for this problem.

The general solution of the y equation is

$$Y(y) = C \sinh \alpha y + D \cosh \alpha y, \tag{5.36}$$

whose derivative (needed in boundary conditions) is

$$Y'(y) = C\alpha \cosh \alpha y + D\alpha \sinh \alpha y. \tag{5.37}$$

The boundary condition $Y'(b) = 0$ implies

$$C\alpha \cosh \alpha b + D\alpha \sinh \alpha b = 0, \tag{5.38}$$

or

$$\frac{C}{D} = -\tanh \alpha b. \tag{5.39}$$

That is, the two constants C and D are not independent of each other. Hence,

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \cos \alpha_n x \, (-\tanh \alpha_n b \sinh \alpha_n y + \cosh \alpha_n y), \tag{5.40}$$

which satisfies the partial differential equation and the three homogeneous boundary conditions. With the fundamental identity

$$\cosh \alpha_n (b - y) = \cosh \alpha_n b \cosh \alpha_n y - \sinh \alpha_n b \sinh \alpha_n y, \tag{5.41}$$

the solution can be written in the alternative forms

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \cos \alpha_n x \, \frac{\cosh \alpha_n (b - y)}{\cosh \alpha_n b} \tag{5.42}$$

or

$$\phi(x, y) = \sum_{n=1}^{\infty} B_n \cos \alpha_n x \, \cosh \alpha_n (b - y). \tag{5.43}$$

The boundary condition at $y = 0$ implies

$$f(x) = \sum_{n=1}^{\infty} A_n \cos \alpha_n x. \tag{5.44}$$

Since the X system (including boundary conditions) is a Sturm-Liouville system, the eigenfunctions $\cos \alpha_n x$ form an orthogonal set. Thus, we can multiply both sides of the last equation by $\cos \alpha_m x$, and integrate from 0 to a, to obtain

$$\int_0^a f(x) \cos \alpha_m x \, dx = A_m \int_0^a \cos^2 \alpha_m x \, dx, \tag{5.45}$$

or

$$A_n = \frac{\int_0^a f(x) \cos \alpha_n x \, dx}{\int_0^a \cos^2 \alpha_n x \, dx}, \tag{5.46}$$

where α_n is the set of solutions of Eq. 5.33. Note that the denominator in Eq. 5.46 is not equal to $a/2$.

We see from this example that the Sturm-Liouville theory provided the necessary orthogonality to evaluate the coefficients.

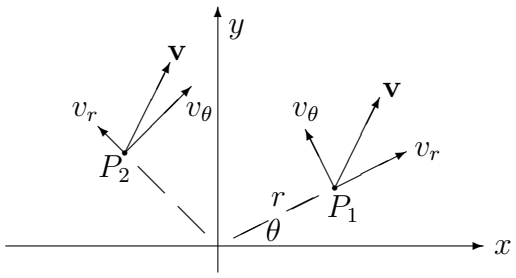


Figure 47: Cylindrical Components of Vector.

6 Orthogonal Curvilinear Coordinates

Consider a two-dimensional vector $\mathbf{v}$ located at Point P_1, as shown in Fig. 47. The *coordinates* of P_1 are (x, y) in Cartesian coordinates and (r, θ) in polar coordinates. The *components* of $\mathbf{v}$ can also be expressed in both coordinate systems. In Cartesian coordinates,

$$\mathbf{v} = v_x \mathbf{e}_x + v_y \mathbf{e}_y, \tag{6.1}$$

where v_x and v_y, the Cartesian components of $\mathbf{v}$, are the projections of $\mathbf{v}$ onto the coordinate directions x and y. Similarly, in polar coordinates, the same vector can be expressed in terms of its polar components:

$$\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta, \tag{6.2}$$

where v_r and v_θ, the components of $\mathbf{v}$ in polar coordinates, are the projections of $\mathbf{v}$ onto the r and θ directions at P_1.

For any coordinate system, a coordinate direction at a point is defined as the initial direction of motion of the point if one coordinate is changed infinitesimally, and all other coordinates are held fixed. For example, the x and y coordinate directions at P_1 are parallel to the x and y axes. The r coordinate direction at P_1 is the radial direction, since that is the direction of movement at P_1 if only r is changed, and θ is held fixed. Similarly, the θ coordinate direction at P_1 is tangential to the circle which passes through P_1 and is centered at the origin. The polar components of $\mathbf{v}$ are thus the projections of $\mathbf{v}$ on the r and θ coordinate directions at P_1. Notice that the basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in polar coordinates have position-dependent orientation. That is, the same vector $\mathbf{v}$ (having the same magnitude and direction) placed at a different location (e.g., P_2 in Fig. 47) has different components in polar coordinates.

We now wish to generalize these ideas to arbitrary orthogonal curvilinear coordinate systems. As we proceed through the theoretical development, it may be useful to use cylindrical coordinates as the example for visualizing the concepts.

Suppose the rectangular coordinates x, y, z are expressed in terms of new coordinates u_1, u_2, u_3 by

$$\begin{cases} x = x(u_1, u_2, u_3) \\ y = y(u_1, u_2, u_3) \\ z = z(u_1, u_2, u_3). \end{cases} \tag{6.3}$$

We assume that we can generally invert these expressions to yield u_1, u_2, u_3 in terms of x, y, z.

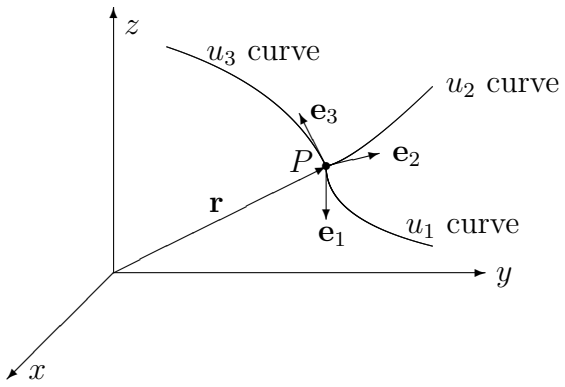


Figure 48: Coordinate Curves in Curvilinear Coordinate System.

If a particle at some point P moves so that only u_1 changes (with u_2 and u_3 held fixed), we generate what is called a u_1 curve, as shown in Fig. 48. The u_2 and u_3 curves are defined similarly. We denote the unit vectors tangent to these three coordinate curves at P as $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. The coordinate system is defined as *orthogonal* if these three unit vectors are everywhere mutually orthogonal. Most of the common coordinate systems used in engineering, including cylindrical and spherical, are orthogonal coordinate systems.

We define the *position vector* $\mathbf{r}$ at point P as

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z. \tag{6.4}$$

The position vector of a point is thus the vector from the coordinate origin to the point. A vector tangent to the u_1 curve is

$$\mathbf{U}_1 = \frac{\partial \mathbf{r}}{\partial u_1}, \tag{6.5}$$

since the numerator represents the infinitesimal *change* in the position vector due to an infinitesimal change in the u_1 coordinate (i.e., we move slightly along the u_1 curve). In general,

$$\mathbf{U}_i = \frac{\partial \mathbf{r}}{\partial u_i} \quad (i = 1, 2, 3). \tag{6.6}$$

This vector is tangent to the u_i curve, and hence parallel to $\mathbf{e}_i$, but not a unit vector in general, since, to be a unit vector, the denominator would have to have the dimension of length. We let h_i denote the scale factor relating the length of the tangent $\mathbf{U}_i$ to that of the unit tangent $\mathbf{e}_i$. (That is, h_i is the length of $\mathbf{U}_i$.) Thus,

$$\mathbf{U}_i = \frac{\partial \mathbf{r}}{\partial u_i} = h_i \mathbf{e}_i \quad \text{(no sum on } i\text{)}, \tag{6.7}$$

where $\mathbf{e}_i$ is the unit vector tangential to the u_i curve, and h_i is the scale factor. That is,

$$h_i = \left| \frac{\partial \mathbf{r}}{\partial u_i} \right| = \left| \frac{\partial x}{\partial u_i}\mathbf{e}_x + \frac{\partial y}{\partial u_i}\mathbf{e}_y + \frac{\partial z}{\partial u_i}\mathbf{e}_z \right|, \tag{6.8}$$

or

$$h_i = \sqrt{\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2}. \tag{6.9}$$

As we will see, the scale factors h_i are fundamental for a given coordinate system, since they allow us to compute all quantities of interest for that system.

6.1 Arc Length and Volume

Let s_i denote arc length along the u_i curve. Then, for the ith coordinate direction,

$$\mathbf{U}_i = \frac{\partial \mathbf{r}}{\partial u_i} = \frac{\partial \mathbf{r}}{\partial s_i}\frac{ds_i}{du_i}, \tag{6.10}$$

where the first factor is the unit vector $\mathbf{e}_i$, and, hence, the second factor is the scale factor h_i Thus,

$$ds_i = h_i\, du_i. \tag{6.11}$$

For example, on a circle of radius r, where arc length $s = r\theta$, $ds = r\, d\theta$. Then, if s is arc length along a curve in any direction (not necessarily a coordinate direction), the chain rule implies

$$\frac{d\mathbf{r}}{ds} = \frac{\partial \mathbf{r}}{\partial u_1}\frac{du_1}{ds} + \frac{\partial \mathbf{r}}{\partial u_2}\frac{du_2}{ds} + \frac{\partial \mathbf{r}}{\partial u_3}\frac{du_3}{ds} = h_1\mathbf{e}_1\frac{du_1}{ds} + h_2\mathbf{e}_2\frac{du_2}{ds} + h_3\mathbf{e}_3\frac{du_3}{ds}. \tag{6.12}$$

However, since $d\mathbf{r}/ds$ is a unit vector,

$$1 = \frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{r}}{ds} = h_1^2\left(\frac{du_1}{ds}\right)^2 + h_2^2\left(\frac{du_2}{ds}\right)^2 + h_3^2\left(\frac{du_3}{ds}\right)^2 \tag{6.13}$$

or

$$ds^2 = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2. \tag{6.14}$$

This is the differential arc length in terms of the scale factors.

The element of volume, defined for an orthogonal coordinate system as $dV = ds_1\, ds_2\, ds_3$, is thus, in terms of the coordinate system scale factors,

$$dV = h_1h_2h_3\, du_1\, du_2\, du_3. \tag{6.15}$$

6.2 Del Operator

Given the scalar function $f(u_1, u_2, u_3)$, recall from §1.2 that the rate of change of f along an arc is

$$\frac{df}{ds} = \nabla f\cdot\frac{d\mathbf{r}}{ds}, \tag{6.16}$$

where the second factor on the right-hand side is a unit vector in the $d\mathbf{r}$ direction. Thus,

$$df = \nabla f\cdot d\mathbf{r}, \tag{6.17}$$

where, from the chain rule, the left-hand side is

$$df = \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2 + \frac{\partial f}{\partial u_3}du_3, \tag{6.18}$$

and $d\mathbf{r}$ is given by Eq. 6.12 as

$$d\mathbf{r} = h_1\mathbf{e}_1\,du_1 + h_2\mathbf{e}_2\,du_2 + h_3\mathbf{e}_3\,du_3. \tag{6.19}$$

We wish to compute ∇f in the form

$$\nabla f = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3, \tag{6.20}$$

where λ_1, λ_2, and λ_3 are to be determined. The substitution of the last three equations into Eq. 6.17 yields

$$\frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2 + \frac{\partial f}{\partial u_3}du_3 = h_1\lambda_1\,du_1 + h_2\lambda_2\,du_2 + h_3\lambda_3\,du_3. \tag{6.21}$$

Since the three du_i are independent,

$$\lambda_i = \frac{1}{h_i}\frac{\partial f}{\partial u_i} \quad \text{(no sum on } i\text{)}, \tag{6.22}$$

and, from Eq. 6.20,

$$\nabla = \mathbf{e}_1\frac{1}{h_1}\frac{\partial}{\partial u_1} + \mathbf{e}_2\frac{1}{h_2}\frac{\partial}{\partial u_2} + \mathbf{e}_3\frac{1}{h_3}\frac{\partial}{\partial u_3}. \tag{6.23}$$

We now use this fundamental result to compute three special cases. From Eq. 6.23,

$$\nabla u_1 = \frac{\mathbf{e}_1}{h_1}, \tag{6.24}$$

or, in general,

$$\nabla u_i = \frac{\mathbf{e}_i}{h_i} \quad (i = 1, 2, 3). \tag{6.25}$$

This result implies

$$\nabla \times \left(\frac{\mathbf{e}_i}{h_i}\right) = \nabla \times \nabla u_i = \mathbf{0}, \tag{6.26}$$

since the curl of the gradient of a scalar field vanishes. To derive the third special case, notice that

$$\nabla \cdot \left(\frac{\mathbf{e}_1}{h_2h_3}\right) = \nabla \cdot \left(\frac{\mathbf{e}_2}{h_2} \times \frac{\mathbf{e}_3}{h_3}\right) = \nabla \cdot (\nabla u_2 \times \nabla u_3) = 0, \tag{6.27}$$

where the last equation follows from another vector identity, Eq. 1.15. Thus, in general,

$$\nabla \cdot \left(\frac{\mathbf{e}_1}{h_2h_3}\right) = \nabla \cdot \left(\frac{\mathbf{e}_2}{h_3h_1}\right) = \nabla \cdot \left(\frac{\mathbf{e}_3}{h_1h_2}\right) = 0. \tag{6.28}$$

We now have the tools to compute several other results of interest.

6.3 Divergence, Laplacian, and Curl

Given a vector field $\mathbf{F}$, we wish to compute the divergence

$$\nabla \cdot \mathbf{F} = \nabla \cdot (F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3) = \nabla \cdot (F_1\mathbf{e}_1) + \nabla \cdot (F_2\mathbf{e}_2) + \nabla \cdot (F_3\mathbf{e}_3). \tag{6.29}$$

The first term, which is typical, is

$$\nabla \cdot (F_1\mathbf{e}_1) = \nabla \cdot \left[(h_2h_3F_1)\left(\frac{\mathbf{e}_1}{h_2h_3}\right)\right] = \nabla(h_2h_3F_1) \cdot \frac{\mathbf{e}_1}{h_2h_3} = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u_1}(h_2h_3F_1), \tag{6.30}$$

where Eqs. 6.28 and 6.23 were used. With a cyclic permutation of the subscripts, we can derive similar expressions for the second and third terms of Eq. 6.29 and obtain

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1) + \frac{\partial}{\partial u_2}(h_3h_1F_2) + \frac{\partial}{\partial u_3}(h_1h_2F_3)\right]. \tag{6.31}$$

To compute the Laplacian $\nabla^2 = \nabla \cdot \nabla$, let $\mathbf{F} = \nabla$ in Eq. 6.31, where ∇ is given by Eq. 6.23, to obtain

$$\nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]. \tag{6.32}$$

Given a vector field $\mathbf{F}$, we wish to compute the curl

$$\nabla \times \mathbf{F} = \nabla \times (F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3) = \nabla \times (F_1\mathbf{e}_1) + \nabla \times (F_2\mathbf{e}_2) + \nabla \times (F_3\mathbf{e}_3). \tag{6.33}$$

The first term, which is typical, is

$$\begin{aligned}\nabla \times (F_1\mathbf{e}_1) &= \nabla \times \left[(h_1F_1)\left(\frac{\mathbf{e}_1}{h_1}\right)\right] = \nabla(h_1F_1) \times \frac{\mathbf{e}_1}{h_1} \\ &= \frac{1}{h_1}\left[\frac{\mathbf{e}_2}{h_3}\frac{\partial}{\partial u_3}(h_1F_1) - \frac{\mathbf{e}_3}{h_2}\frac{\partial}{\partial u_2}(h_1F_1)\right] \\ &= \frac{1}{h_1h_2h_3}\left[h_2\mathbf{e}_2\frac{\partial}{\partial u_3}(h_1F_1) - h_3\mathbf{e}_3\frac{\partial}{\partial u_2}(h_1F_1)\right],\end{aligned} \tag{6.34}$$

where Eq. 6.23 was used. With a cyclic permutation of the subscripts, we can derive similar expressions for the second and third terms of Eq. 6.33, and express the final result in the form of a determinant:

$$\nabla \times \mathbf{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}. \tag{6.35}$$

Notice that, for the special case of Cartesian coordinates,

$$(u_1, u_2, u_3) = (x, y, z) \qquad \text{and} \qquad h_1 = h_2 = h_3 = 1, \tag{6.36}$$

and all results of this section reduce to the expected results.

6.4 Example: Cylindrical Coordinates

We illustrate the equations of this chapter for cylindrical coordinates $(u_1, u_2, u_3) = (r, \theta, z)$, defined as

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ z = z. \end{cases} \tag{6.37}$$

To avoid confusion between the coordinate r and the usual notation for the position vector, we let $\mathbf{R}$ denote the position vector for this example:

$$\mathbf{R} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = (r\cos\theta)\mathbf{e}_x + (r\sin\theta)\mathbf{e}_y + z\mathbf{e}_z. \tag{6.38}$$

From Eq. 6.6,

$$\begin{cases} \mathbf{U}_r = \partial\mathbf{R}/\partial r = \cos\theta\,\mathbf{e}_x + \sin\theta\,\mathbf{e}_y \\ \mathbf{U}_\theta = \partial\mathbf{R}/\partial\theta = (-r\sin\theta)\mathbf{e}_x + (r\cos\theta)\mathbf{e}_y \\ \mathbf{U}_z = \partial\mathbf{R}/\partial z = \mathbf{e}_z. \end{cases} \tag{6.39}$$

We observe that the cylindrical coordinate system is an orthogonal coordinate system, since, for all (r, θ, z),

$$\mathbf{U}_r \cdot \mathbf{U}_\theta = \mathbf{U}_\theta \cdot \mathbf{U}_z = \mathbf{U}_z \cdot \mathbf{U}_r = 0. \tag{6.40}$$

Hence, from Eq. 6.39, the scale factors are

$$\begin{cases} h_r = |\mathbf{U}_r| = \sqrt{\cos^2\theta + \sin^2\theta} = 1 \\ h_\theta = |\mathbf{U}_\theta| = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = r \\ h_z = |\mathbf{U}_z| = 1, \end{cases} \tag{6.41}$$

and, from Eq. 6.7,

$$\begin{cases} \mathbf{e}_r = \quad \cos\theta\ \mathbf{e}_x + \sin\theta\ \mathbf{e}_y \\ \mathbf{e}_\theta = -\sin\theta\ \mathbf{e}_x + \cos\theta\ \mathbf{e}_y \\ \mathbf{e}_z = \mathbf{e}_z. \end{cases} \tag{6.42}$$

Given the scale factors, we then obtain

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2, \qquad \text{(arc length)} \tag{6.43}$$

$$dV = r\,dr\,d\theta\,dz, \qquad \text{(element of volume)} \tag{6.44}$$

$$\nabla\phi = \mathbf{e}_r\frac{\partial\phi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\phi}{\partial\theta} + \mathbf{e}_z\frac{\partial\phi}{\partial z}, \qquad \text{(gradient)} \tag{6.45}$$

$$\begin{aligned} \nabla\cdot\mathbf{F} &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_r) + \frac{\partial}{\partial\theta}(F_\theta) + \frac{\partial}{\partial z}(rF_z)\right] \\ &= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}, \qquad \text{(divergence)} \end{aligned} \tag{6.46}$$

$$\nabla^2\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\phi}{\partial z}\right)\right]$$
$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}, \qquad \text{(Laplacian)} \tag{6.47}$$

and

$$\nabla\times\mathbf{F} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}. \qquad \text{(curl)} \tag{6.48}$$

7 Problems in Cylindrical Coordinates

7.1 Bessel's Equation

In boundary value problems that involve the Laplacian ∇^2 in cylindrical coordinates, the process of separation of variables often produces an ordinary differential equation of the form

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - \nu^2)y = 0 \quad (\nu \text{ specified}), \tag{7.1}$$

or, in Sturm-Liouville form,

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(\lambda^2x - \frac{\nu^2}{x}\right)y = 0, \tag{7.2}$$

which implies $p(x) = x$, $q(x) = -\nu^2/x$, $r(x) = x$, and the eigenvalue is λ^2. A more common form results after the change of variable $r = \lambda x$:

$$r^2y''(r) + ry'(r) + (r^2 - \nu^2)y(r) = 0, \tag{7.3}$$

which is Bessel's Equation of order ν. Since ν appears only as ν^2 in this equation, we may consider ν to be nonnegative without loss of generality.

Linear ordinary differential equations having polynomial coefficients can often be solved conveniently using a power series method called the method of Frobenius [10]. For Bessel's Equation, Eq. 7.3, we assume a solution of the form

$$y = r^p\sum_{i=0}^{\infty}a_ir^i = \sum_{i=0}^{\infty}a_ir^{p+i}, \tag{7.4}$$

where p will be chosen so that a_0 represents the coefficient of the first nonzero term in the series.

The substitution of this series into Eq. 7.3 yields

$$\sum_{i=0}^{\infty}\left[r^2a_i(p+i)(p+i-1)r^{p+i-2} + ra_i(p+i)r^{p+i-1} + (r^2-\nu^2)a_ir^{p+i}\right] = 0. \tag{7.5}$$

We factor this equation to obtain

$$\sum_{i=0}^{\infty} \left[(p+i)(p+i-1) + (p+i) + (r^2 - \nu^2)\right] a_i r^{p+i} = 0, \tag{7.6}$$

or

$$\sum_{i=0}^{\infty} \left[(p+i)^2 - \nu^2\right] a_i r^{p+i} + \sum_{i=0}^{\infty} a_i r^{p+i+2} = 0. \tag{7.7}$$

We now divide every term in the series by r^p, and replace i with $i-2$ in the second series, to obtain

$$\sum_{i=0}^{\infty} \left[(p+i)^2 - \nu^2\right] a_i r^{i} + \sum_{i=2}^{\infty} a_{i-2} r^{i} = 0. \tag{7.8}$$

Since the ranges of the two summations differ, we extract the first two terms from the first sum, after which the remainder of the first sum can be combined with the second sum:

$$(p^2 - \nu^2)a_0 + [(p+1)^2 - \nu^2]a_1 r + \sum_{i=2}^{\infty} \left\{ \left[(p+i)^2 - \nu^2\right] a_i + a_{i-2} \right\} r^i = 0. \tag{7.9}$$

This equation must be satisfied identically, in the sense that the coefficients of all powers of r must vanish independently. That is, for this equation to hold for all r, each coefficient must vanish. The vanishing of the coefficient of the zero term implies that $p = \pm\nu$ or $a_0 = 0$. The vanishing of the coefficient of the first term implies that $p+1 = \pm\nu$ or $a_1 = 0$. Since we (arbitrarily) choose $a_0 \neq 0$, it follows that $p = \pm\nu$ and $a_1 = 0$ (since, if $p = \pm\nu$, $p+1 \neq \pm\nu$). The vanishing of the coefficient of the ith term implies

$$a_i = \frac{-a_{i-2}}{(p+i+\nu)(p+i-\nu)}. \tag{7.10}$$

To decide whether $p = +\nu$ or $p = -\nu$, notice that, if $p = +\nu$,

$$a_i = \frac{-a_{i-2}}{i(2\nu+i)}, \quad i = 2, 3, \ldots, \tag{7.11}$$

whereas, if $p = -\nu$,

$$a_i = \frac{-a_{i-2}}{i(-2\nu+i)}, \quad i = 2, 3, \ldots. \tag{7.12}$$

Since the denominator in Eq. 7.12 vanishes for $i = 2\nu$, and the denominator in Eq. 7.11 never vanishes for $i = 2, 3, \ldots$, we choose $p = +\nu$ ($\nu \geq 0$), so that the coefficients a_i are given by Eq. 7.11. Since $a_1 = 0$, it follows that $a_3 = a_5 = a_7 = \cdots = 0$.

Had we chosen $a_0 = 0$ and $p+1 = \nu$, the even terms would vanish instead of the odd terms, but, since p would have been replaced with $p-1$, the resulting series would have been identical.

Since only the even coefficients are nonzero, we define $i = 2k$ ($k = 0, 1, 2, 3, \ldots$). Hence, we have the recurrence formula

$$a_{2k} = \frac{-a_{2k-2}}{4k(\nu+k)}, \quad k = 1, 2, 3, \ldots. \tag{7.13}$$

That is,

$$a_2 = \frac{-a_0}{4 \cdot 1(\nu+1)}, \tag{7.14}$$

$$a_4 = \frac{-a_2}{4 \cdot 2(\nu+2)} = \frac{a_0}{4 \cdot 4 \cdot 1 \cdot 2(\nu+1)(\nu+2)}, \tag{7.15}$$

$$a_6 = \frac{-a_4}{4 \cdot 3(\nu+3)} = \frac{-a_0}{4 \cdot 4 \cdot 4 \cdot 1 \cdot 2 \cdot 3(\nu+1)(\nu+2)(\nu+3)}, \tag{7.16}$$

or, in general,

$$a_{2k} = \frac{(-1)^k a_0}{4^k k!(\nu+1)(\nu+2)\cdots(\nu+k)}. \tag{7.17}$$

Since the differential equation, Eq. 7.3, is homogeneous, and every coefficient in the series is proportional to a_0, a_0 is arbitrary, and we choose

$$a_0 = \frac{1}{\nu! 2^\nu}, \tag{7.18}$$

since this choice results in a more convenient form for the coefficients,

$$a_{2k} = \frac{(-1)^k}{2^{\nu+2k} k!(\nu+k)!}, \tag{7.19}$$

and the series solution of Bessel's Equation, Eq. 7.3, is

$$y(r) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{\nu+2k} k!(\nu+k)!} r^{\nu+2k}. \tag{7.20}$$

We denote this solution $J_\nu(r)$; i.e.,

$$J_\nu(r) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(\nu+i)!} \left(\frac{r}{2}\right)^{\nu+2i}, \tag{7.21}$$

which is called Bessel's Function of the First Kind of order ν. In particular, for $\nu = 0$ and $\nu = 1$,

$$J_0(r) = 1 - \left(\frac{r}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{r}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{r}{2}\right)^6 + \cdots, \tag{7.22}$$

and

$$J_1(r) = \frac{r}{2} - \frac{1}{2!}\left(\frac{r}{2}\right)^3 + \frac{1}{2!3!}\left(\frac{r}{2}\right)^5 - \frac{1}{3!4!}\left(\frac{r}{2}\right)^7 + \cdots. \tag{7.23}$$

We note that $J_0'(r) = -J_1(r)$.

We have used the notation $(\nu+i)!$ without requiring that ν be an integer. Although our interest in this volume is in integer values of ν, ν may, in general, be non-integer, in which case $J_\nu(r)$ is usually written

$$J_\nu(r) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,\Gamma(\nu+i+1)} \left(\frac{r}{2}\right)^{\nu+2i}, \tag{7.24}$$

where the gamma function $\Gamma(x)$ is defined [22] as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \quad (x > 0). \tag{7.25}$$

This function is the unique extension of the factorial function to non-integer arguments, since, according to the Bohr-Mollerup theorem [2], it is the only function $f(x)$ defined for $0 < x < \infty$ with the properties

1. $\log f(x)$ is convex (concave up)
2. $f(x+1) = xf(x)$ for all $x > 0$, and
3. $f(1) = 1$.

For the gamma function, the second of these properties follows by integration by parts. Thus, for positive integer arguments,

$$\Gamma(n) = (n-1)! \quad (n = 1, 2, 3, \ldots). \tag{7.26}$$

Since $J_\nu(r)$ is a solution of Eq. 7.3, $J_\nu(\lambda x)$ is a solution of Eq. 7.1. If boundary conditions of Sturm-Liouville type are specified at $x = a$ and $x = b$, the orthogonality relationship for $J_\nu(\lambda x)$ is, according to Eq. 5.9,

$$\int_a^b x J_\nu(\lambda x) J_\nu(\bar{\lambda} x)\, dx = 0 \quad \text{for } \lambda \neq \bar{\lambda}, \tag{7.27}$$

where λ and $\bar{\lambda}$ are two distinct eigenvalues, and the order ν is fixed.

Since Bessel's Equation is second order, there are two linearly independent solutions, the first of which is $J_\nu(r)$ in Eq. 7.21. For integer orders, the second solution is denoted $Y_\nu(r)$, which is called the Bessel Function of the Second Kind of order ν.

The general solution of Bessel's Equation is thus

$$y(r) = c_1 J_\nu(r) + c_2 Y_\nu(r), \tag{7.28}$$

where c_1 and c_2 are constants.

There are many ways of expressing Bessel functions, including integral forms. For $\nu = n$ an integer ($n = 0, 1, 2, \ldots$), it can be shown that

$$J_n(r) = \frac{1}{\pi} \int_0^\pi \cos(n\phi - r \sin\phi)\, d\phi. \tag{7.29}$$

This relation, which can be proved by direct substitution into Bessel's Equation, allows us to derive some useful properties of Bessel functions.

To show the boundedness of $J_n(r)$, we note that, since the integrand in Eq. 7.29 is bounded, in absolute value, by unity, it follows that

$$|J_n(r)| \leq \frac{1}{\pi} \int_0^\pi 1\, d\phi = 1. \tag{7.30}$$

Similarly, the integrands in

$$J_n'(r) = \frac{1}{\pi}\int_0^\pi \sin(n\phi - r\sin\phi)\sin\phi\, d\phi \tag{7.31}$$

and

$$J_n''(r) = -\frac{1}{\pi}\int_0^\pi \cos(n\phi - r\sin\phi)\sin^2\phi\, d\phi \tag{7.32}$$

are also bounded, in absolute value, by unity, implying that

$$|J_n'(r)| \le \frac{1}{\pi}\int_0^\pi 1\, d\phi = 1 \tag{7.33}$$

and

$$|J_n''(r)| \le 1. \tag{7.34}$$

Thus, in general,

$$|J_n^{(k)}(r)| \le 1 \quad (-\infty < r < \infty, \quad k = 0, 1, 2, 3, \ldots). \tag{7.35}$$

To determine the qualitative behavior of Bessel functions near the origin, we first observe that, from Eq. 7.29,

$$J_n(0) = \frac{1}{\pi}\int_0^\pi \cos n\phi\, d\phi, \tag{7.36}$$

which implies

$$J_n(0) = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases} \tag{7.37}$$

These results are consistent with Eqs. 7.22 and 7.23.

Bessel's Equation, Eq. 7.3, for $\nu = n$, can be written in the form

$$y''(r) + \frac{1}{r}y'(r) + \left(1 - \frac{n^2}{r^2}\right)y(r) = 0. \tag{7.38}$$

For $n \neq 0$ and small r, the Bessel functions behave like the solutions of

$$y'' + \frac{1}{r}y' - \frac{n^2}{r^2}y = 0 \tag{7.39}$$

or

$$r^2y'' + ry' - n^2y = 0. \tag{7.40}$$

This equation, which is characterized by terms in which the power of r matches the order of the derivative, is called Euler's differential equation. This equation has solutions of the form $y = r^m$, which, when substituted in the differential equation, yields

$$0 = r^2m(m-1)r^{m-2} + rmr^{m-1} - n^2r^m = r^m(m^2 - n^2). \tag{7.41}$$

Hence, $m = \pm n$, and, near $r = 0$, the two solutions of Bessel's Equation behave like r^n and r^{-n}. Since $r^n \to 0$ as $r \to 0$, this solution corresponds to $J_n(r)$. Since the second solution $r^{-n} \to \infty$ as $r \to 0$, we conclude that $Y_n(r) \to \infty$ as $r \to 0$.

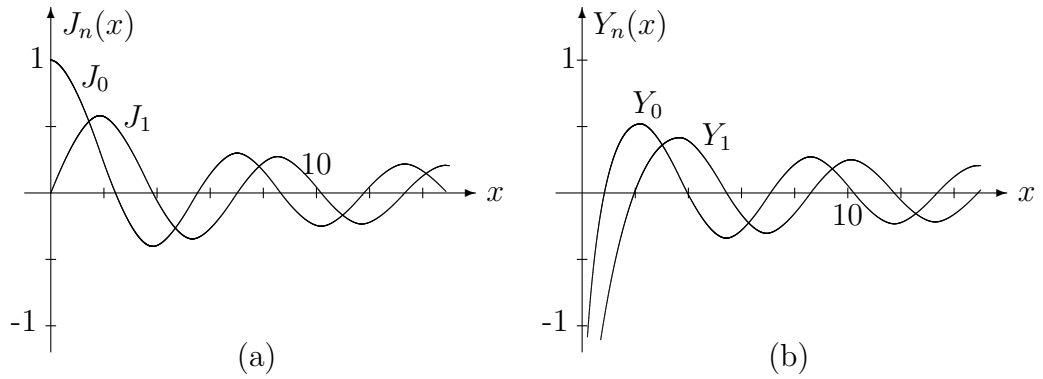


Figure 49: Plots of Some Bessel Functions.

The qualitative behavior of Bessel functions for large argument can be determined with a change of variable. The change of variable $y = u/\sqrt{r}$ applied to Bessel's Equation, Eq. 7.38, results in the equivalent equation

$$u'' + \left(\frac{1/4 - n^2}{r^2} + 1 \right) u = 0. \tag{7.42}$$

To derive this result, we substitute the derivatives

$$\frac{dy}{dr} = \frac{1}{\sqrt{r}} \frac{du}{dr} - \frac{1}{2} u r^{-3/2} \tag{7.43}$$

and

$$\frac{d^2y}{dr^2} = \frac{1}{\sqrt{r}} u'' - r^{-3/2} u' + \frac{3}{4} r^{-5/2} u \tag{7.44}$$

into Eq. 7.38.

For large r, the solutions of Eq. 7.42 behave like those of

$$u'' + u = 0, \tag{7.45}$$

whose solutions are $\sin r$ and $\cos r$. Thus, the solutions of Bessel's Equation, for large r, behave like $\sin r/\sqrt{r}$ and $\cos r/\sqrt{r}$. That is, both Bessel functions $J_n(r)$ and $Y_n(r)$ behave like decaying sinusoids for large r. Plots of several Bessel functions are shown in Fig. 49.

7.2 Example

Consider the following two-dimensional problem, where the goal is to find $T(r, \theta, t)$ in the interior of the circle of radius a:

$$\begin{cases} \nabla^2 T + \dfrac{Q}{k} = \dfrac{1}{K} \dfrac{\partial T}{\partial t}, \\ T(a, \theta, t) = f(\theta), \\ T(r, \theta, 0) = T_0(r, \theta), \end{cases} \tag{7.46}$$

where $Q = Q(r, \theta)$ (Fig. 50). In two dimensions in cylindrical coordinates,

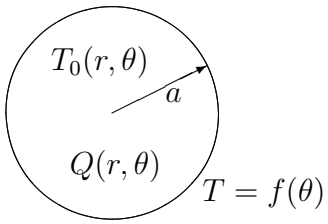


Figure 50: Problem in 2-D Cylindrical Coordinates.

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}. \tag{7.47}$$

This problem is similar to that of §4.2 except for the change in shape of the domain.

The problem in Eq. 7.46 has three forcing functions: the internal source $Q(r,\theta)$, the nonzero boundary condition $f(\theta)$, and the initial condition $T_0(r,\theta)$. We therefore attempt a solution as the sum of three functions, one for each forcing function:

$$T(r,\theta,t) = u(r,\theta,t) + v_1(r,\theta) + v_2(r,\theta). \tag{7.48}$$

We intend for u to be the transient part of the solution, v_1 to have the source Q as its forcing function, and v_2 to have the nonzero boundary condition f as its forcing function. The substitution of Eq. 7.48 into Eq. 7.46 yields

$$\begin{cases} \nabla^2 u + \nabla^2 v_1 + \nabla^2 v_2 + \dfrac{Q}{k} = \dfrac{1}{K}\dfrac{\partial u}{\partial t}, \\ u(a,\theta,t) + v_1(a,\theta) + v_2(a,\theta) = f(\theta), \\ u(r,\theta,0) + v_1(r,\theta) + v_2(r,\theta) = T_0(r,\theta). \end{cases} \tag{7.49}$$

The original problem can thus be decomposed into the three problems

$$\text{I.} \begin{cases} \nabla^2 u = \dfrac{1}{K}\dfrac{\partial u}{\partial t}, \\ u(a,\theta,t) = 0, \\ u(r,\theta,0) = T_0(r,\theta) - v_1(r,\theta) - v_2(r,\theta), \end{cases} \tag{7.50}$$

$$\text{II.} \begin{cases} \nabla^2 v_1 = -Q/k, \\ v_1(a,\theta) = 0, \end{cases} \tag{7.51}$$

$$\text{III.} \begin{cases} \nabla^2 v_2 = 0, \\ v_2(a,\theta) = f(\theta). \end{cases} \tag{7.52}$$

Notice that, if the initial condition $T_0(r,\theta)$ were zero, there would still be a transient part of the problem, since the initial condition for the transient part of the problem, Eq. 7.50c, is the difference between the original initial condition and the steady-state solution. We now consider each of these three problems in turn.

7.2.1 The Transient Problem

We attempt a solution of Eq. 7.50, a homogeneous equation, using separation of variables with, at first, a separation between space and time,

$$u(r,\theta,t) = w(r,\theta)\mathcal{T}(t), \tag{7.53}$$

which, when substituted into Eq. 7.50a yields

$$(\nabla^2 w)\mathcal{T} = \frac{1}{K} w \mathcal{T}' \tag{7.54}$$

or

$$\frac{\nabla^2 w}{w} = \frac{1}{K}\frac{\mathcal{T}'}{\mathcal{T}} = -\alpha^2, \tag{7.55}$$

where the sign of the separation constant α^2 was selected because we anticipate that $\mathcal{T}(t)$ will decay in time. Thus, the separated equations are

$$\begin{cases} \mathcal{T}' + K\alpha^2 \mathcal{T} = 0, \\ \nabla^2 w + \alpha^2 w = 0. \end{cases} \tag{7.56}$$

The time equation, Eq. 7.56a, has as its solution

$$\mathcal{T}(t) = e^{-K\alpha^2 t}. \tag{7.57}$$

The space equation, Eq. 7.56b, is the Helmholtz equation, Eq. 2.18. Once again, like in Cartesian coordinates, we obtain the Helmholtz equation for the spacial part of a transient heat conduction problem. In two-dimensional cylindrical coordinates, Eq. 7.56b is

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \alpha^2 w = 0. \tag{7.58}$$

The solution w can be further separated into the product

$$w(r, \theta) = R(r)\Theta(\theta), \tag{7.59}$$

which, when substituted into Eq. 7.58 yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + \alpha^2 R\Theta = 0, \tag{7.60}$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \alpha^2 r^2 = -\frac{\Theta''}{\Theta} = \beta^2, \tag{7.61}$$

where the sign of the second separation constant β^2 was chosen to yield a Θ solution which is periodic in θ. Thus, the separated ordinary differential equations are

$$\begin{cases} \Theta'' + \beta^2\Theta = 0, \\ r^2 R'' + rR' + (\alpha^2 r^2 - \beta^2)R = 0. \end{cases} \tag{7.62}$$

The general solution of the Θ equation, Eq. 7.62a, is

$$\Theta(\theta) = A\sin\beta\theta + B\cos\beta\theta. \tag{7.63}$$

Since the circular domain extends from 0 to 2π, the only "boundary conditions" we have to impose on Θ are those of periodicity, which requires that

$$\begin{cases} \Theta(\theta) = \Theta(\theta + 2\pi), \\ \Theta'(\theta) = \Theta'(\theta + 2\pi) \end{cases} \tag{7.64}$$

for all θ. The second condition, Eq. 7.64b, is needed to avoid slope discontinuities (kinks) in the Θ solution. Eqs. 7.64 imply that β must be an integer ($\beta = m,\ m = 0, 1, 2, \ldots$), and hence

$$\Theta(\theta) = A \sin m\theta + B \cos m\theta. \tag{7.65}$$

For $m = 0$, this solution no longer involves two independent functions, so we must solve the original ordinary differential equation again for this case. For $\beta = m = 0$, $\Theta'' = 0$ implies $\Theta(\theta) = E\theta + F$, where periodicity requires $E = 0$. Hence, for $m = 0$, $\Theta(\theta) = 1$.

Notice also that the periodicity conditions are sufficient to guarantee orthogonality of the Θ solution, since, in the Sturm-Liouville theory, Eq. 5.8 (p. 57) is applicable to Eq. 7.62a if

$$a = \theta,\ b = \theta + 2\pi,\ \lambda = \beta^2,\ p(x) = 1,\ r(x) = 1. \tag{7.66}$$

Thus, if the eigenfunction and its first derivative are periodic, and $p(a) = p(b)$, we have orthogonality for the eigenfunctions of Eq. 7.62a. This system is called a *periodic Sturm-Liouville system.*

The R equation, Eq. 7.62b, with $\beta = m$ is Bessel's equation of order m,

$$r^2 R'' + rR' + (\alpha^2 r^2 - m^2)R = 0, \tag{7.67}$$

whose general solution is

$$R(r) = CJ_m(\alpha r) + DY_m(\alpha r). \tag{7.68}$$

The requirement that T, and hence u, be finite at $r = 0$ yields $D = 0$, since Y_m, the Bessel Function of the Second Kind, is singular at the origin. The boundary condition $u(a, \theta, t) = 0$ implies

$$0 = R(a) = CJ_m(\alpha a). \tag{7.69}$$

Since Bessel functions are oscillatory functions, this equation yields an infinite set of α solutions for each m. In general, α_{mn} satisfies

$$J_m(\alpha_{mn} a) = 0, \tag{7.70}$$

where $\alpha_{mn} a$ is the nth zero of the mth order Bessel function.

The product of the three separated solutions,

$$J_m(\alpha_{mn} r)(A \sin m\theta + B \cos m\theta)e^{-K\alpha_{mn}^2 t},$$

satisfies, for each m and n, the partial differential equation, periodicity, finiteness at the origin, and the boundary condition. Hence, the general solution is

$$u(r, \theta, t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} J_m(\alpha_{mn} r)(A_{mn} \sin m\theta + B_{mn} \cos m\theta)e^{-K\alpha_{mn}^2 t}. \tag{7.71}$$

The remaining condition, the initial condition, requires

$$T_0 - v_1 - v_2 = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} J_m(\alpha_{mn} r)(A_{mn} \sin m\theta + B_{mn} \cos m\theta). \tag{7.72}$$

To evaluate the coefficients, we recall the orthogonality relation for the Bessel functions,

$$\int_0^a r J_m(\alpha_{mn} r) J_m(\alpha_{m\bar{n}} r)\, dr = 0 \quad \text{for } n \neq \bar{n}, \tag{7.73}$$

where the eigenvalue is n, and the order m is fixed. We now multiply both sides of Eq. 7.72 by $\sin \bar{m}\theta$, and integrate from 0 to 2π, to obtain

$$\int_0^{2\pi} (T_0 - v_1 - v_2) \sin \bar{m}\theta \, d\theta = \sum_{n=1}^{\infty} J_{\bar{m}}(\alpha_{\bar{m}n} r) A_{\bar{m}n} \pi \quad (\bar{m} > 0). \tag{7.74}$$

If we then multiply this result by $r J_{\bar{m}}(\alpha_{\bar{m}\bar{n}} r)$, and integrate from 0 to a, we obtain

$$\int_0^a r J_{\bar{m}}(\alpha_{\bar{m}\bar{n}} r) \left[\int_0^{2\pi} (T_0 - v_1 - v_2) \sin \bar{m}\theta \, d\theta \right] dr = A_{\bar{m}\bar{n}} \pi \int_0^a r [J_{\bar{m}}(\alpha_{\bar{m}\bar{n}} r)]^2 \, dr, \tag{7.75}$$

or

$$A_{mn} = \frac{\int_0^a r J_m(\alpha_{mn} r) \left[\int_0^{2\pi} (T_0 - v_1 - v_2) \sin m\theta \, d\theta \right] dr}{\pi \int_0^a r [J_m(\alpha_{mn} r)]^2 \, dr} \quad (m > 0). \tag{7.76}$$

B_{mn} is found similarly, except $\sin m\theta$ is replaced with $\cos m\theta$ in the above expression:

$$B_{mn} = \frac{\int_0^a r J_m(\alpha_{mn} r) \left[\int_0^{2\pi} (T_0 - v_1 - v_2) \cos m\theta \, d\theta \right] dr}{\pi \int_0^a r [J_m(\alpha_{mn} r)]^2 \, dr} \quad (m > 0). \tag{7.77}$$

For $m = 0$, π in the denominator is replaced with 2π. Therefore, the solution of the u equation is the series, Eq. 7.71, with the coefficients given by Eqs. 7.76 and 7.77.

7.2.2 The Steady-State Problem with Nonzero Source

The solution approach for the v_1 equation, Eq. 7.51, is similar to that used in rectangular coordinates in §4.2.2. From the u solution of the preceding section, we recall that the product

$$J_m(\alpha_{mn} r)(C_{mn} \sin m\theta + D_{mn} \cos m\theta)$$

satisfies the boundary condition (i.e., the function is zero at $r = a$), is finite at the origin and periodic, and can be used as the basis for a generalized two-dimensional Fourier series expansion in the r and θ directions. Thus, we look for a solution of Eq. 7.51 in the form

$$v_1(r, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} J_m(\alpha_{mn} r)(C_{mn} \sin m\theta + D_{mn} \cos m\theta). \tag{7.78}$$

Since each term of this function satisfies the Helmholtz equation, Eq. 7.56b, the Laplacian $\nabla^2 v_1$ can be easily computed. (In fact, the motivation in the preceding section for first separating the u equation into space and time variables was to observe that the spacial part of the solution satisfies the Helmholtz equation, thus providing a mechanism for solving nonhomogeneous partial differential equations involving the Laplacian.)

We substitute Eq. 7.78 into Eq. 7.51a to obtain

$$\nabla^2 v_1 = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} -\alpha_{mn}^2 J_m(\alpha_{mn} r)(C_{mn}\sin m\theta + D_{mn}\cos m\theta) = -Q/k. \tag{7.79}$$

Orthogonality in the r and θ solutions then yields

$$C_{mn}\alpha_{mn}^2 \pi \int_0^a r[J_m(\alpha_{mn} r)]^2\, dr = \int_0^{2\pi}\int_0^a [Q/k] J_m(\alpha_{mn} r)\sin m\theta\, r\, dr\, d\theta \tag{7.80}$$

or

$$C_{mn} = \frac{\int_0^{2\pi}\int_0^a [Q/k] J_m(\alpha_{mn} r)\sin m\theta\, r\, dr\, d\theta}{\alpha_{mn}^2 \pi \int_0^a r[J_m(\alpha_{mn} r)]^2\, dr}, \tag{7.81}$$

where the set $\{\alpha_{mn} a,\ n = 1, 2, 3, \ldots\}$ are the zeros of the Bessel function of order m. The constants D_{mn} are found similarly if $\sin m\theta$ above is replaced with $\cos m\theta$:

$$D_{mn} = \frac{\int_0^{2\pi}\int_0^a [Q/k] J_m(\alpha_{mn} r)\cos m\theta\, r\, dr\, d\theta}{\alpha_{mn}^2 \pi \int_0^a r[J_m(\alpha_{mn} r)]^2\, dr} \quad (m > 0). \tag{7.82}$$

For $m = 0$, π in the denominator is replaced with 2π.

In general the source Q is a function of position. For the special case $Q =$ constant, we note from the last two equations that $C_{mn} = 0$ for all m, and $D_{mn} = 0$ for $m > 0$. However, in this case, there is no need to expand the solution in a Fourier series involving zero order Bessel functions, since Eq. 7.51 can be solved in closed form by direct integration, since, if $Q =$ constant, the solution has rotational symmetry.

7.2.3 The Steady-State Problem with Nonzero Boundary Data

The forcing function for Problem III (the v_2 equation, Eq. 7.52) appears in the nonzero boundary data. Since the partial differential equation is homogeneous, we attempt a solution using the technique of separation of variables. We note that the Laplacian in two-dimensional cylindrical coordinates is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{7.83}$$

The substitution

$$v_2(r,\theta) = R(r)\Theta(\theta) \tag{7.84}$$

in Eq. 7.52a yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \tag{7.85}$$

or

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \alpha^2, \tag{7.86}$$

where the sign of the separation constant α^2 was selected to yield periodic solutions in the θ direction. The separated ordinary differential equations are thus

$$\begin{cases} r^2 R'' + rR' - \alpha^2 R = 0, \\ \Theta'' + \alpha^2\Theta = 0. \end{cases} \tag{7.87}$$

The general solution of the Θ equation is

$$\Theta = A \sin \alpha\theta + B \cos \alpha\theta. \tag{7.88}$$

Periodicity in the θ direction requires

$$\begin{cases} \Theta(\theta) = \Theta(\theta + 2\pi), \\ \Theta'(\theta) = \Theta'(\theta + 2\pi) \end{cases} \tag{7.89}$$

for all θ, which implies α must be an integer ($\alpha = m,\ m = 0, 1, 2, \ldots$), and hence

$$\Theta(\theta) = A \sin m\theta + B \cos m\theta \quad (m > 0). \tag{7.90}$$

The R equation, Eq. 7.87a, is Euler's equation, which is characterized by terms in which the power of r matches the order of the derivative. Solutions are of the form $R = r^s$, which, when substituted in the differential equation, yields

$$0 = s(s-1)r^2 r^{s-2} + srr^{s-1} - m^2 r^s = r^s(s^2 - s + s - m^2) = r^s(s^2 - m^2). \tag{7.91}$$

Hence, $s = \pm m$, and

$$R(r) = Cr^m + Dr^{-m}, \tag{7.92}$$

where finiteness of the solution at the origin implies $D = 0$. The product of the R and Θ solutions is thus

$$v_2(r, \theta) = r^m(A \sin m\theta + B \cos m\theta) \quad (m > 0). \tag{7.93}$$

To address the special case $\alpha = m = 0$, we must solve the two separated ordinary differential equations again, since the solutions found for $m > 0$ are not independent when $m = 0$. For $m = 0$, the separated equations are

$$\begin{cases} r^2 R'' + rR' = r\dfrac{d}{dr}(rR') = 0, \\ \Theta'' = 0. \end{cases} \tag{7.94}$$

whose solutions are, respectively,

$$\begin{cases} R(r) = c_1 \ln r + c_2, \\ \Theta(\theta) = b_1 + b_2\theta. \end{cases} \tag{7.95}$$

Periodicity implies $b_2 = 0$, and finiteness at the origin implies $c_1 = 0$. Thus, for $m = 0$, $v_2(r, \theta)$ is a constant. The combination of this solution with the sum of the solutions for all $m > 0$, Eq. 7.93, yields

$$v_2(r, \theta) = B_0 + \sum_{m=1}^{\infty} r^m(A_m \sin m\theta + B_m \cos m\theta), \tag{7.96}$$

which satisfies the partial differential equation, periodicity, and finiteness at the origin.

The boundary condition at $r = a$, Eq. 7.52b, implies

$$f(\theta) = B_0 + \sum_{m=1}^{\infty} a^m (A_m \sin m\theta + B_m \cos m\theta). \tag{7.97}$$

To determine A_m, we multiply both sides by $\sin \bar{m}\theta$, and integrate from 0 to 2π to obtain

$$\int_0^{2\pi} f(\theta) \sin \bar{m}\theta \, d\theta = a^{\bar{m}} A_{\bar{m}} \int_0^{2\pi} \sin^2 \bar{m}\theta \, d\theta = a^{\bar{m}} A_{\bar{m}} \pi \tag{7.98}$$

or

$$A_m = \frac{1}{\pi a^m} \int_0^{2\pi} f(\theta) \sin m\theta \, d\theta. \tag{7.99}$$

Similarly, for B_m, we replace $\sin m\theta$ with $\cos m\theta$ to obtain

$$B_m = \frac{1}{\pi a^m} \int_0^{2\pi} f(\theta) \cos m\theta \, d\theta \quad (m > 0). \tag{7.100}$$

For $m = 0$, we integrate both sides of Eq. 7.97 from 0 to 2π to obtain

$$\int_0^{2\pi} f(\theta) \, d\theta = B_0 \int_0^{2\pi} d\theta = B_0(2\pi) \tag{7.101}$$

or

$$B_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \, d\theta. \tag{7.102}$$

Notice that the expression for B_m, Eq. 7.100, applies also to $m = 0$ if π in the denominator is replaced with 2π.

8 Problems in Spherical Coordinates

8.1 Legendre's Equation

In boundary/initial value problems that involve the Laplacian ∇^2 in spherical coordinates, the process of separation of variables often produces an ordinary differential equation of the form

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \quad (-1 \le x \le 1), \tag{8.1}$$

or, in Sturm-Liouville form,

$$\frac{d}{dx}[(1 - x^2)y'] + \lambda y = 0, \tag{8.2}$$

with $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$, and the eigenvalue is λ. This equation is called Legendre's equation.

We solve Eq. 8.1 using the Frobenius method, and look for solutions of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{8.3}$$

The substitution of this series into Eq. 8.1 yields

$$\sum_{k=0}^{\infty}[(1-x^2)k(k-1)a_k x^{k-2} - 2xka_k x^{k-1} + \lambda a_k x^k] = 0 \tag{8.4}$$

or, if we combine like powers of x,

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=0}^{\infty}[-k(k+1)+\lambda]a_k x^k = 0, \tag{8.5}$$

where the lower limit of the first series was changed from $k=0$ to $k=2$, since the first two terms (corresponding to $k=0$ and $k=1$) are zero. We now replace k with $k+2$ in the first series of this equation to obtain

$$\sum_{k=0}^{\infty}\{(k+2)(k+1)a_{k+2} - [k(k+1)-\lambda]a_k\}x^k = 0. \tag{8.6}$$

For this equation to be an identity for all x, each term must vanish identically, and

$$a_{k+2} = \frac{k(k+1)-\lambda}{(k+2)(k+1)}\, a_k, \tag{8.7}$$

which is a recursion relation giving the even coefficients in terms of a_0 and the odd coefficients in terms of a_1. Thus, a_0 and a_1 are the only arbitrary coefficients.

We now write λ as

$$\lambda = n(n+1), \tag{8.8}$$

since, given λ, there exists an n such that Eq. 8.8 holds. With this substitution, Eq. 8.7 becomes

$$a_{k+2} = \frac{k(k+1)-n(n+1)}{(k+2)(k+1)}\, a_k. \tag{8.9}$$

Since k, the index of the original series, is an integer, it follows that, if n is also an integer, the numerator of Eq. 8.9 vanishes for $k=n$, and the series terminates, since, in that case, $a_{n+2}=0$ and, by Eq. 8.7,

$$a_{n+4} = a_{n+6} = \cdots = 0. \tag{8.10}$$

We assume that n is an integer and will see in the discussion following Eq. 8.36 that this assumption is correct. Since, under this assumption, the series terminates, the solution is not a series, but a polynomial of degree n. To write all coefficients in terms of the highest coefficient, we first invert Eq. 8.9 to obtain

$$a_k = \frac{(k+2)(k+1)}{k(k+1)-n(n+1)}\, a_{k+2} = -\frac{(k+2)(k+1)}{(n-k)(n+k+1)}\, a_{k+2}. \tag{8.11}$$

In particular,

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)}\, a_n, \tag{8.12}$$

$$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)}\, a_{n-2} = \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}\, a_n, \tag{8.13}$$

and

$$a_{n-6} = -\frac{(n-4)(n-5)}{6(2n-5)}\, a_{n-4} = -\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 4\cdot 6(2n-1)(2n-3)(2n-5)}\, a_n. \tag{8.14}$$

In general,

$$a_{n-2k} = \frac{(-1)^k}{2^k k!} \cdot \frac{n(n-1)(n-2)\cdots(n-2k+1)}{(2n-1)(2n-3)\cdots(2n-2k+1)}\, a_n. \tag{8.15}$$

We now multiply both numerator and denominator of this equation by

$$(2n-2k-1)(2n-2k-3)\cdots(1)\cdot(n-2k)!$$

to obtain $n!$ in the numerator and the product of all odd integers up to $2n-1$ in the denominator, and Eq. 8.15 becomes

$$a_{n-2k} = \frac{(-1)^k}{2^k k!} \cdot \frac{n!}{(2n-1)(2n-3)\cdots(1)} \cdot \frac{(2n-2k-1)(2n-2k-3)\cdots(1)}{(n-2k)!}\, a_n. \tag{8.16}$$

We next multiply both numerator and denominator of Eq. 8.16 by $(2n-2k)(2n-2k-2)\cdots(2)$, which is the product of all even integers up to $2n-2k$, to obtain

$$a_{n-2k} = \frac{(-1)^k}{2^k k!} \cdot \frac{n!}{(2n-1)(2n-3)\cdots(1)} \cdot \frac{(2n-2k)!}{(n-2k)!\, 2^{n-k}(n-k)!}\, a_n. \tag{8.17}$$

Since a_n is arbitrary, we choose it such that

$$\frac{n!}{(2n-1)(2n-3)\cdots(1)}\, a_n = 1, \tag{8.18}$$

so that

$$a_{n-2k} = \frac{(-1)^k}{2^n k!} \cdot \frac{(2n-2k)!}{(n-2k)!\,(n-k)!}. \tag{8.19}$$

We can write the solution, which is a polynomial of degree n, in the form

$$y(x) = \sum_{k=0}^{m} a_{n-2k}\, x^{n-2k}, \tag{8.20}$$

where a_{n-2k} is given by Eq. 8.19, and m is chosen so that $n-2k$ in the last term (the lowest order term) is 1 for n odd or 0 for n even. Thus,

$$m = \begin{cases} n/2, & n \text{ even}, \\ (n-1)/2, & n \text{ odd}. \end{cases} \tag{8.21}$$

This polynomial is called Legendre's polynomial of degree n and denoted P_n:

$$P_n(x) = \sum_{k=0}^{m} \frac{(-1)^k (2n-2k)!}{2^n k!(n-2k)!\,(n-k)!}\, x^{n-2k}, \tag{8.22}$$

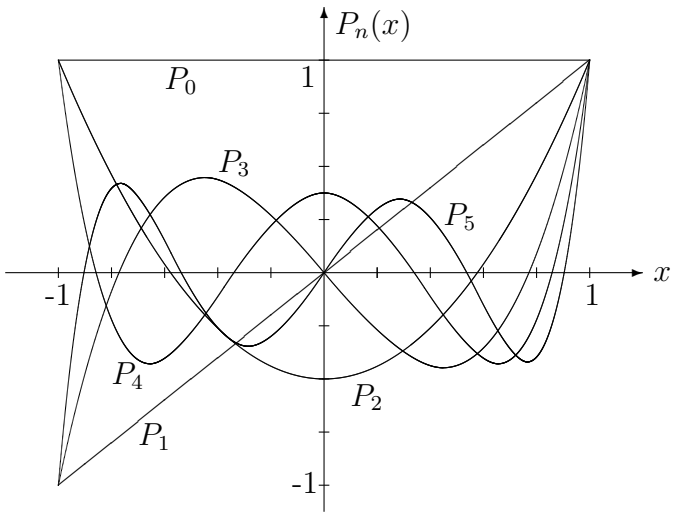


Figure 51: Plots of Some Legendre Polynomials.

where n is related to λ of Eq. 8.1 by $\lambda = n(n+1)$. The necessity of this relationship will be shown later in this section in the discussion following Eq. 8.36. Note that $P_n(x)$ has only even or odd terms.

Direct evaluation of this formula, Eq. 8.22, yields

$$P_0(x) = 1, \tag{8.23}$$

$$P_1(x) = x, \tag{8.24}$$

$$P_2(x) = (3x^2 - 1)/2, \tag{8.25}$$

$$P_3(x) = (5x^3 - 3x)/2, \tag{8.26}$$

$$P_4(x) = (35x^4 - 30x^2 + 3)/8, \tag{8.27}$$

$$P_5(x) = (63x^5 - 70x^3 + 15x)/8, \tag{8.28}$$

$$P_6(x) = (231x^6 - 315x^4 + 105x^2 - 5)/16, \tag{8.29}$$

$$P_7(x) = (429x^7 - 693x^5 + 315x^3 - 35x)/16. \tag{8.30}$$

Notice from Eqs. 8.23–8.30 that

$$P_n(1) = 1, \quad P_n(-1) = (-1)^n. \tag{8.31}$$

These results turn out to be general results for all n. It can also be shown that Legendre polynomials satisfy the recurrence formula

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x), \quad n = 2, 3, 4, \ldots, \tag{8.32}$$

which is useful for numerical evaluation of the polynomials. Plots of the first six Legendre polynomials are shown in Fig. 51.

The polynomial solution $P_n(x)$ found above is one solution of Legendre's equation. Since the differential equation is second order, there exists a second independent solution, denoted

$Q_n(x)$, which is called Legendre's function of the second kind. Thus, the general solution of Legendre's equation is

$$y(x) = CP_n(x) + DQ_n(x), \tag{8.33}$$

where C and D are constants. Since it turns out that Q_n is singular at $x = \pm 1$, our only interest in this volume is with the first solution, $P_n(x)$.

The Legendre polynomials $P_n(x)$ satisfy Legendre's equation

$$\frac{d}{dx}[(1-x^2)P_n'] + \lambda_n P_n = 0, \tag{8.34}$$

which is a Sturm-Liouville equation with $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$, and

$$\lambda_n = n(n+1). \tag{8.35}$$

With the domain of interest $(-1, 1)$ and $p(\pm 1) = 0$, the Legendre polynomials thus satisfy the orthogonality relation

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \text{ for } m \neq n \tag{8.36}$$

according to Remark 1 on p. 58.

Since a Taylor series is complete (i.e., all smooth functions can be represented with a Taylor series), the set $\{P_n\}$, an orthogonal set which includes polynomials of all orders, is also complete. Thus, the only possible eigenvalues of Legendre's equation are those given by Eq. 8.35, since, if there were any other eigenvalue, the corresponding eigenfunction would have to be orthogonal to *all* $\{P_n\}$. However, since $\{P_n\}$ is a complete set, there can be no function orthogonal to the entire set, and eigenvalues other than those given by Eq. 8.35 do not exist. Thus, the assumption following Eq. 8.9 that n is an integer is also verified, since there can be no other solutions of Legendre's equation.

To compute the norm of Legendre polynomials, we wish to evaluate

$$(P_n, P_n) = \int_{-1}^{1} [P_n(x)]^2\,dx \tag{8.37}$$

for different n. By direct substitution of the polynomials in Eqs. 8.23–8.30 into Eq. 8.37, we obtain

$$(P_0, P_0) = 2\int_0^1 1\,dx = 2/1, \tag{8.38}$$

$$(P_1, P_1) = 2\int_0^1 x^2\,dx = 2/3, \tag{8.39}$$

$$(P_2, P_2) = 2\int_0^1 \frac{1}{4}(3x^2-1)^2\,dx = 2/5, \tag{8.40}$$

$$(P_3, P_3) = 2\int_0^1 \frac{1}{4}(5x^3-3x)^2\,dx = 2/7, \tag{8.41}$$

from which we conclude that, in general,

$$\int_{-1}^{1} [P_n(x)]^2\,dx = \frac{2}{2n+1}. \tag{8.42}$$

It can be shown that this equation holds for all n.

8.2 Interior Domains

We define spherical coordinates by the relations

$$\begin{cases} x = r\sin\theta\cos\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\theta. \end{cases} \tag{8.43}$$

We refer to θ, the angle measured from the North pole, as the colatitude, and ϕ as the longitude. For this coordinate system, the scale factors defined in Eq. 6.9 are

$$\begin{cases} h_r = 1, \\ h_\theta = r, \\ h_\phi = r\sin\theta, \end{cases} \tag{8.44}$$

and the Laplacian, from Eq. 6.32, is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{8.45}$$

Consider steady-state heat conduction in the interior of a homogeneous solid sphere with the temperature prescribed on the outer boundary to be a function of the colatitude θ only:

$$\begin{cases} \nabla^2 T = 0 \quad (r < a), \\ T = f(\theta) \text{ at } r = a, \end{cases} \tag{8.46}$$

where the Laplacian is given by Eq. 8.45. Since the only forcing function, $f(\theta)$, is independent of ϕ, the solution $T(r,\theta)$ is also independent of ϕ, and Eq. 8.46a simplifies to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial T}{\partial\theta}\right) = 0. \tag{8.47}$$

We attempt a separation of variables solution

$$T(r,\theta) = R(r)\Theta(\theta) \tag{8.48}$$

which, when substituted into Eq. 8.47, yields

$$\frac{\partial}{\partial r}(r^2R'\Theta) + \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}(\sin\theta\cdot R\Theta') = 0, \tag{8.49}$$

or

$$\frac{1}{R}\frac{d}{dr}(r^2R') = -\frac{1}{\Theta\sin\theta}\,\frac{d}{d\theta}(\sin\theta\cdot\Theta') = \lambda, \tag{8.50}$$

where λ is the separation constant. For this problem, there is no easy way to anticipate the correct sign for λ. Thus, the separated equations are

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}(\sin\theta\cdot\Theta') + \lambda\Theta = 0, \tag{8.51}$$

$$\frac{d}{dr}(r^2R') - \lambda R = 0. \tag{8.52}$$

In the Θ equation, Eq. 8.51, the change of variable $x = \cos\theta$ yields

$$\frac{d}{dx}\left[(1-x^2)\frac{d\Theta}{dx}\right] + \lambda\Theta = 0 \quad (-1 \le x \le 1), \tag{8.53}$$

since

$$dx = -\sin\theta\, d\theta \tag{8.54}$$

and

$$\sin\theta \cdot \Theta' = \frac{1-\cos^2\theta}{\sin\theta}\,\frac{d\Theta}{d\theta} = -(1-x^2)\frac{d\Theta}{dx}. \tag{8.55}$$

Note that the new variable x introduced here is not the Cartesian coordinate x used in Eq. 8.43a. Eq. 8.53 is Legendre's equation, where, according to the discussion following Eq. 8.36, the eigenvalue λ is necessarily given by

$$\lambda = n(n+1), \quad n = 0, 1, 2, \ldots \tag{8.56}$$

and, hence,

$$\Theta(\theta) = P_n(\cos\theta). \tag{8.57}$$

Since $Q_n(\cos\theta)$, the Legendre function of the second kind, is singular for $x = \cos\theta = \pm 1$, Q_n is of no interest.

The R equation, Eq. 8.52, can be expanded to yield

$$r^2R'' + 2rR' - \lambda R = 0. \tag{8.58}$$

This equation, which is characterized by terms in which the power of r matches the order of the derivative, is called Euler's differential equation. Solutions are of the form $R = r^s$, which, when substituted in the differential equation, yields

$$0 = r^2 s(s-1)r^{s-2} + 2rsr^{s-1} - \lambda r^s = r^s[s(s-1) + 2s - \lambda] = r^s(s^2 + s - \lambda), \tag{8.59}$$

where λ is given by Eq. 8.56. Thus,

$$s = \frac{-1 \pm \sqrt{1+4\lambda}}{2} = \begin{cases} n, \\ -n-1, \end{cases} \tag{8.60}$$

and

$$R(r) = C_1 r^n + \frac{C_2}{r^{n+1}}, \quad n = 0, 1, 2, \ldots, \tag{8.61}$$

where C_1 and C_2 are constants. Since the domain for this problem is the *interior* of the sphere, the solution T, and hence R, must be finite at $r = 0$, implying $C_2 = 0$.

The combination of the R and Θ solutions yields the general solution

$$T(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta), \tag{8.62}$$

which satisfies the partial differential equation, Eq. 8.46a, and finiteness at the origin. It is convenient to redefine the coefficients A_n, and write this equation in the nondimensional form

$$T(r,\theta) = \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{8.63}$$

The boundary condition $T(a,\theta) = f(\theta)$ requires

$$f(\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta) \tag{8.64}$$

or

$$f(\cos^{-1} x) = \sum_{n=0}^{\infty} A_n P_n(x), \tag{8.65}$$

which is a form more convenient for applying orthogonality, Eqs. 8.36 and 8.42. If we multiply both sides of this equation by $P_m(x)$, and integrate from -1 to 1, we obtain

$$\int_{-1}^{1} f(\cos^{-1} x) P_m(x)\, dx = A_m \left(\frac{2}{2m+1}\right) \tag{8.66}$$

or

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} f(\cos^{-1} x) P_n(x)\, dx. \tag{8.67}$$

8.3 Exterior Domains

We now consider the problem of the preceding section with an exterior, rather than interior, domain:

$$\begin{cases} \nabla^2 T = 0 \quad (r > a), \\ T(a,\theta) = f(\theta), \\ T(\infty,\theta) = 0, \end{cases} \tag{8.68}$$

where we require that the solution at infinity be zero.

The separation of variables solution for this problem is identical to that of the preceding section and yields the same result:

$$\Theta(\theta) = P_n(\cos\theta), \tag{8.69}$$

$$R(r) = C_1 r^n + \frac{C_2}{r^{n+1}}, \quad n = 0, 1, 2, \ldots . \tag{8.70}$$

In contrast to the interior problem of the preceding section, finiteness of the solution at infinity requires $C_1 = 0$. Thus, the general solution is, in nondimensional form,

$$T(r,\theta) = \sum_{n=0}^{\infty} A_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta), \tag{8.71}$$

where the boundary condition $T(a,\theta) = f(\theta)$ yields Eq. 8.64, the same equation obtained for the interior problem in the preceding section. Thus, the coefficients for the exterior problem are also given by Eq. 8.67.

9 Fourier Integral Transforms

9.1 Fourier Integral Theorem and Fourier Transforms

Recall from §3.5 that a function $f(x)$ defined in $-L \le x \le L$ has the Fourier Series representation

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \tag{9.1}$$

where

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(\xi)\, d\xi, \tag{9.2}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos\frac{n\pi\xi}{L}\, d\xi, \tag{9.3}$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\sin\frac{n\pi\xi}{L}\, d\xi, \tag{9.4}$$

where we use ξ to denote the dummy variable of integration. If we substitute these coefficients into the series, Eq. 9.1, we obtain

$$f(x) = A_0 + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(\xi)\left[\cos\frac{n\pi\xi}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi\xi}{L}\sin\frac{n\pi x}{L}\right] d\xi \tag{9.5}$$

or

$$f(x) = A_0 + \sum_{n=1}^{\infty}\frac{1}{L}\int_{-L}^{L} f(\xi)\cos\frac{n\pi}{L}(\xi - x)\, d\xi. \tag{9.6}$$

For functions $f(x)$ defined instead in $-\infty < x < \infty$, we wish to consider the effect on Eq. 9.6 of taking the limit $L \to \infty$. To gain some insight into this question, consider first the function $g(\alpha)$, defined for $0 < \alpha < \infty$, for which we want to evaluate the area G under the curve:

$$G = \int_0^{\infty} g(\alpha)\, d\alpha. \tag{9.7}$$

A discretization of the α axis into subintervals of width $\Delta\alpha$ yields an approximate method (Fig. 52) to evaluate G:

$$G \approx g(\alpha_1)\Delta\alpha + g(\alpha_2)\Delta\alpha + g(\alpha_3)\Delta\alpha + \cdots = \sum_{n=1}^{\infty} g(\alpha_n)\Delta\alpha, \tag{9.8}$$

where $\alpha_n = n\Delta\alpha$.

In the Fourier series, Eq. 9.6, we let

$$\alpha_n = \frac{n\pi}{L}, \quad \Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{\pi}{L}, \tag{9.9}$$

so that

$$f(x) = A_0 + \sum_{n=1}^{\infty}\frac{\Delta\alpha}{\pi}\int_{-\pi/\Delta\alpha}^{\pi/\Delta\alpha} f(\xi)\cos\alpha_n(\xi - x)\, d\xi = A_0 + \frac{1}{\pi}\sum_{n=1}^{\infty} g(\alpha_n)\Delta\alpha, \tag{9.10}$$

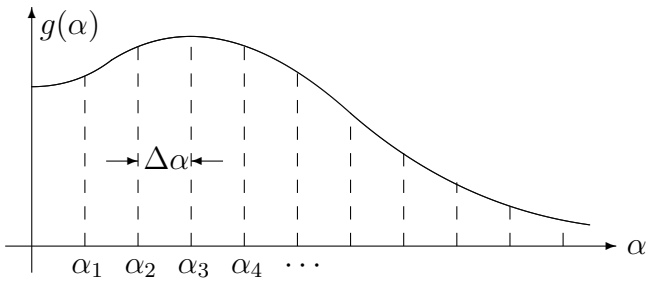


Figure 52: Approximate Integration.

where

$$g(\alpha_n) = \int_{-\pi/\Delta\alpha}^{\pi/\Delta\alpha} f(\xi)\cos\alpha_n(\xi - x)\,d\xi. \tag{9.11}$$

We now take the limit $L \to \infty$, which implies $\Delta\alpha \to 0$ and

$$f(x) = A_0 + \frac{1}{\pi}\int_0^\infty g(\alpha)\,d\alpha = A_0 + \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(\xi)\cos\alpha(\xi - x)\,d\xi\,d\alpha. \tag{9.12}$$

To be assured that A_0, Eq. 9.2, is finite as $L \to \infty$, we assume that $f(x)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^\infty |f(x)|\,dx < M \tag{9.13}$$

for some finite number M. With this restriction, $A_0 \to 0$ as $L \to \infty$, and

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(\xi)\cos\alpha(\xi - x)\,d\xi\,d\alpha, \tag{9.14}$$

which is referred to as the real form of the Fourier Integral Theorem.

The function

$$\int_{-\infty}^\infty f(\xi)\sin\alpha(\xi - x)\,d\xi$$

is an odd function of α, so that

$$\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(\xi)\sin\alpha(\xi - x)\,d\xi\right] d\alpha = 0. \tag{9.15}$$

Also, the function

$$\int_{-\infty}^\infty f(\xi)\cos\alpha(\xi - x)\,d\xi$$

is an even function of α, so that

$$\int_0^\infty \left[\int_{-\infty}^\infty f(\xi)\cos\alpha(\xi - x)\,d\xi\right] d\alpha = \frac{1}{2}\int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(\xi)\cos\alpha(\xi - x)\,d\xi\right] d\alpha, \tag{9.16}$$

which implies, from Eq. 9.14, that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(\xi)\cos\alpha(\xi - x)\,d\xi\,d\alpha + \frac{i}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(\xi)\sin\alpha(\xi - x)\,d\xi\,d\alpha, \tag{9.17}$$

where the second term vanishes according to Eq. 9.15, and $i = \sqrt{-1}$. Thus,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi)[\cos\alpha(\xi - x) + i \sin\alpha(\xi - x)]\, d\xi\, d\alpha \tag{9.18}$$

or

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi) e^{i\alpha(\xi - x)}\, d\xi\, d\alpha, \tag{9.19}$$

which is referred to as the complex form of the Fourier Integral Theorem.

The complex form of the Fourier Integral Theorem, Eq. 9.19, can be written in the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x}\, d\alpha \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{i\alpha\xi}\, d\xi. \tag{9.20}$$

If we let $\bar{f}(\alpha)$ denote the second integral, the result is the *Fourier Transform pair*

$$\bar{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\, dx, \tag{9.21}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha) e^{-i\alpha x}\, d\alpha, \tag{9.22}$$

where $\bar{f}(\alpha)$ is called the *Fourier transform* of $f(x)$, and $f(x)$ is the *inverse Fourier transform* of $\bar{f}(\alpha)$. The dummy variable of integration in Eq. 9.21 was changed to x.

The real form of the Fourier Integral Theorem, Eq. 9.14, can be expanded in the form

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} [f(\xi)\cos\alpha\xi\cos\alpha x + f(\xi)\sin\alpha\xi\sin\alpha x]\, d\xi\, d\alpha. \tag{9.23}$$

If f is even, the two terms of the integrand are, respectively, even and odd functions of ξ, in which case

$$\begin{aligned} f(x) &= \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} f(\xi)\cos\alpha\xi\cos\alpha x\, d\xi\, d\alpha \\ &= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos\alpha x\, d\alpha \cdot \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(\xi)\cos\alpha\xi\, d\xi. \end{aligned} \tag{9.24}$$

If we let $\bar{f}_c(\alpha)$ denote the second integral, the result is the *Fourier Cosine Transform pair*

$$\bar{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x)\cos\alpha x\, dx, \tag{9.25}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \bar{f}_c(\alpha)\cos\alpha x\, d\alpha, \tag{9.26}$$

where the dummy variable of integration in the first equation was changed to x. These formulas are applicable for even functions on $(-\infty, \infty)$ or those functions defined only on $(0, \infty)$.

Similarly, if $f(x)$ is odd or defined only on $(0, \infty)$, we obtain the *Fourier Sine Transform pair*:

$$\bar{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x)\sin\alpha x\, dx, \tag{9.27}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \bar{f}_s(\alpha)\sin\alpha x\, d\alpha. \tag{9.28}$$

9.2 Application to Integral Evaluation

Although our main interest in Fourier transforms is in the solution of partial differential equations, we illustrate here the application of Fourier transforms to the evaluation of integrals. For example, consider the function

$$f(x) = e^{-bx}, \quad 0 \le x < \infty \quad (b > 0). \tag{9.29}$$

To make f an even function, we extend it to negative x such that

$$f(x) = e^{-b|x|}, \quad -\infty < x < \infty \quad (b > 0). \tag{9.30}$$

If this function is substituted into the Fourier cosine transform, Eq. 9.25, we obtain

$$\bar{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \cos \alpha x \, dx, \tag{9.31}$$

which can be evaluated using successive applications of integration by parts. Since

$$d(uv) = (du)v + u(dv), \tag{9.32}$$

integration by parts states that

$$\int_a^b u \, dv = (uv)\Big|_a^b - \int_a^b v \, du, \tag{9.33}$$

where the first term of the right-hand side is the boundary term. Thus, in Eq. 9.31, we make the associations

$$u = e^{-bx}, \quad dv = \cos \alpha x \, dx \tag{9.34}$$

to obtain

$$\sqrt{\frac{\pi}{2}} \, \bar{f}_c(\alpha) = e^{-bx} \frac{\sin \alpha x}{\alpha}\bigg|_0^\infty + \int_0^\infty b e^{-bx} \, \frac{\sin \alpha x}{\alpha} \, dx, \tag{9.35}$$

where the boundary term (the first term on the right-hand side) is zero. We use integration by parts again, with the associations

$$u = e^{-bx}, \quad dv = \frac{\sin \alpha x}{\alpha} \, dx, \tag{9.36}$$

to obtain

$$\sqrt{\frac{\pi}{2}} \, \bar{f}_c(\alpha) = \frac{b}{\alpha} \left(\frac{-e^{-bx} \cos \alpha x}{\alpha} \right)\bigg|_0^\infty - \int_0^\infty \frac{b}{\alpha} \, b e^{-bx} \, \frac{\cos \alpha x}{\alpha} \, dx \tag{9.37}$$

or

$$\sqrt{\frac{\pi}{2}} \, \bar{f}_c(\alpha) = \frac{b}{\alpha^2} - \frac{b^2}{\alpha^2} \int_0^\infty e^{-bx} \cos \alpha x \, dx = \frac{b}{\alpha^2} - \frac{b^2}{\alpha^2} \sqrt{\frac{\pi}{2}} \, \bar{f}_c(\alpha). \tag{9.38}$$

Since $\bar{f}_c(\alpha)$ appears on both sides of this equation,

$$\bar{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \, \frac{b}{\alpha^2 + b^2}. \tag{9.39}$$

On the other hand, from the inverse Fourier cosine transform, Eq. 9.26,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{b}{\alpha^2 + b^2} \cos \alpha x \, d\alpha, \tag{9.40}$$

where $f(x)$ is given by Eq. 9.29. Hence,

$$e^{-bx} = \frac{2}{\pi} \int_0^\infty \frac{b}{\alpha^2 + b^2} \cos \alpha x \, d\alpha. \tag{9.41}$$

This last result shows one example of how transform pairs can be used to evaluate integrals.

For a second example, we combine Eqs. 9.31 and 9.39 to obtain

$$\int_0^\infty e^{-bx} \cos \alpha x \, dx = \frac{b}{\alpha^2 + b^2}. \tag{9.42}$$

If both sides of this equation are integrated from $\alpha = 0$ to $\alpha = \bar{\alpha}$, we obtain

$$\int_0^\infty e^{-bx} \frac{\sin \bar{\alpha} x}{x} \, dx = \tan^{-1} \frac{\bar{\alpha}}{b}. \tag{9.43}$$

We now take the limit $b \to 0$, and replace $\bar{\alpha}$ with α, to obtain

$$\int_0^\infty \frac{\sin \alpha x}{x} \, dx = \frac{\pi}{2}. \tag{9.44}$$

9.3 Operational Table

To apply Fourier transforms to partial differential equations, we need to know how derivatives of functions transform. Since we denoted the Fourier transform of $f(x)$ by $\bar{f}(\alpha)$, we denote the Fourier transform of $f'(x)$ by $\bar{f}^{(1)}(\alpha)$. In general, the Fourier transform of $f^{(n)}(x)$, the nth derivative of f, is denoted $\bar{f}^{(n)}(\alpha)$.

From the exponential form of the Fourier transform, Eq. 9.21,

$$\bar{f}^{(1)}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \frac{df}{dx} e^{i\alpha x} \, dx, \tag{9.45}$$

which can be integrated by parts, Eq. 9.33, with the associations

$$u = e^{i\alpha x}, \quad dv = \frac{df}{dx} dx = df. \tag{9.46}$$

Thus,

$$\bar{f}^{(1)}(\alpha) = \frac{1}{\sqrt{2\pi}} \left[f e^{i\alpha x} \Big|_{-\infty}^{\infty} - i\alpha \int_{-\infty}^\infty f e^{i\alpha x} \, dx \right]. \tag{9.47}$$

The boundary term on the right-hand side (the first term) is an oscillating function of x for large x. Since we restricted f to be absolutely integrable, Eq. 9.13, it follows that

$$f(x) \to 0 \text{ as } x \to \pm\infty, \tag{9.48}$$

implying that the boundary term vanishes. Hence,

$$\bar{f}^{(1)}(\alpha) = -\frac{1}{\sqrt{2\pi}}\, i\alpha \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\, dx \tag{9.49}$$

or

$$\bar{f}^{(1)}(\alpha) = -i\alpha \bar{f}(\alpha). \tag{9.50}$$

In general,

$$\bar{f}^{(n)}(\alpha) = (-i\alpha)^n \bar{f}(\alpha), \tag{9.51}$$

where "(n)" means the nth derivative. That is, given the Fourier transform of f, we are able to evaluate the transform of any derivative of f. Note, however, for this result to hold, f and its first $n-1$ derivatives must vanish at $x = \pm\infty$.

Similar results can be derived for the Fourier cosine and sine transforms. The Fourier cosine transform of $f'(x)$ is, from Eq. 9.25,

$$\bar{f}_c^{(1)}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f'(x) \cos \alpha x\, dx. \tag{9.52}$$

Using integration by parts, with the associations

$$u = \cos \alpha x, \quad dv = f'(x)dx = df, \tag{9.53}$$

we obtain

$$\bar{f}_c^{(1)}(\alpha) = \sqrt{\frac{2}{\pi}} \left[f \cos \alpha x \Big|_0^{\infty} + \alpha \int_0^{\infty} f \sin \alpha x\, dx \right], \tag{9.54}$$

where the integral is $\bar{f}_s(\alpha)$. Hence, since $f(\infty) = 0$,

$$\bar{f}_c^{(1)}(\alpha) = \alpha \bar{f}_s(\alpha) - \sqrt{\frac{2}{\pi}}\, f(0). \tag{9.55}$$

For the sine transform,

$$\bar{f}_s^{(1)}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f'(x) \sin \alpha x\, dx. \tag{9.56}$$

Using integration by parts, with the associations

$$u = \sin \alpha x, \quad dv = f'(x)dx = df, \tag{9.57}$$

we obtain

$$\bar{f}_s^{(1)}(\alpha) = \sqrt{\frac{2}{\pi}} \left[f \sin \alpha x \Big|_0^{\infty} - \alpha \int_0^{\infty} f \cos \alpha x\, dx \right], \tag{9.58}$$

where the integral is $\bar{f}_c(\alpha)$, and the boundary term vanishes. Hence,

$$\bar{f}_s^{(1)}(\alpha) = -\alpha \bar{f}_c(\alpha). \tag{9.59}$$

Formulas for second derivatives can be derived by successive application of these formulas, Eqs. 9.55 and 9.59:

$$\bar{f}_c^{(2)}(\alpha) = \alpha \bar{f}_s^{(1)}(\alpha) - \sqrt{\frac{2}{\pi}}\, f'(0) = -\alpha^2 \bar{f}_c(\alpha) - \sqrt{\frac{2}{\pi}}\, f'(0). \tag{9.60}$$

Also,

$$\bar{f}_s^{(2)}(\alpha) = -\alpha \bar{f}_c^{(1)}(\alpha) = -\alpha^2 \bar{f}_s(\alpha) + \sqrt{\frac{2}{\pi}}\, \alpha f(0). \tag{9.61}$$

Eqs. 9.50, 9.51, 9.55, 9.59, 9.60, and 9.61 constitute our operational table, since they can be used to transform any of the partial differential equations encountered so far. For convenience, we collect all these results together:

$$\bar{f}^{(1)}(\alpha) = -i\alpha \bar{f}(\alpha), \tag{9.62}$$

$$\bar{f}^{(n)}(\alpha) = (-i\alpha)^n \bar{f}(\alpha), \tag{9.63}$$

$$\bar{f}_c^{(1)}(\alpha) = \alpha \bar{f}_s(\alpha) - \sqrt{\frac{2}{\pi}}\, f(0), \tag{9.64}$$

$$\bar{f}_s^{(1)}(\alpha) = -\alpha \bar{f}_c(\alpha), \tag{9.65}$$

$$\bar{f}_c^{(2)}(\alpha) = -\alpha^2 \bar{f}_c(\alpha) - \sqrt{\frac{2}{\pi}}\, f'(0), \tag{9.66}$$

$$\bar{f}_s^{(2)}(\alpha) = -\alpha^2 \bar{f}_s(\alpha) + \sqrt{\frac{2}{\pi}}\, \alpha f(0). \tag{9.67}$$

9.4 Example 1

Consider the semi-infinite one-dimensional transient heat conduction problem

$$\begin{cases} \dfrac{\partial^2 T}{\partial x^2} = \dfrac{\partial T}{\partial t}, \quad x > 0, \\ T(0,t) = T_0, \quad \lim\limits_{x\to\infty} T(x,t) = 0, \\ T(x,0) = 0, \end{cases} \tag{9.68}$$

where T_0 is a constant, and $T = T(x,t)$. That is, for the right-half space initially at zero temperature, we wish to compute the time history of temperature if the temperature of the boundary at $x = 0$ is suddenly fixed at the constant temperature T_0 (Fig. 53).

There are three possible Fourier transforms from which to choose for solving this problem: exponential, cosine, and sine. The exponential transform cannot be used, since the domain for this problem is $(0, \infty)$, and the exponential transform requires that T be defined over $(-\infty, \infty)$. The cosine transform cannot be used, since, from the operational table, Eq. 9.66, taking the cosine transform of Eq. 9.68a requires the knowledge of the *derivative* of temperature at $x = 0$. On the other hand, from Eq. 9.67, we can take the sine transform of Eq. 9.68a, since knowledge of only the function is required at $x = 0$.

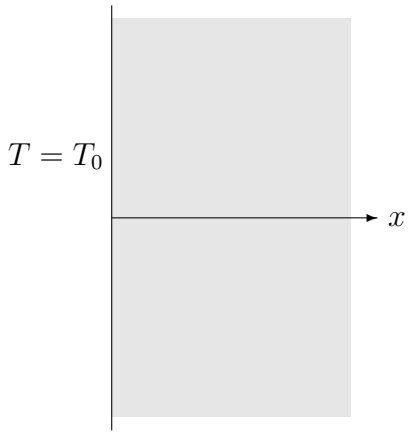


Figure 53: Right-Half Space With Dirichlet B.C.

The Fourier sine transform of the unknown function $T(x,t)$ is

$$\bar{T}(\alpha,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty T(x,t) \sin \alpha x \, dx. \tag{9.69}$$

Then, from the operational table,

$$\begin{aligned} T(x,t) &\to \bar{T}(\alpha,t), \\ \frac{\partial^2 T(x,t)}{\partial x^2} &\to \sqrt{\frac{2}{\pi}}\, \alpha\, T(0,t) - \alpha^2 \bar{T}(\alpha,t), \\ \frac{\partial T(x,t)}{\partial t} &\to \frac{\partial \bar{T}(\alpha,t)}{\partial t}, \end{aligned} \tag{9.70}$$

where here the "$\to$" symbol means "transforms to."

If we take the Fourier sine transform of Eq. 9.68a, we obtain

$$\alpha \sqrt{\frac{2}{\pi}}\, T(0,t) - \alpha^2 \bar{T}(\alpha,t) = \frac{\partial \bar{T}(\alpha,t)}{\partial t}, \tag{9.71}$$

where $T(0,t) = T_0$. We can treat α like a parameter in this equation, since the equation is an ordinary differential equation in time. Thus,

$$\frac{d\bar{T}}{dt} + \alpha^2 \bar{T} = \alpha \sqrt{\frac{2}{\pi}}\, T_0, \tag{9.72}$$

whose solution is the sum of the solution of the corresponding homogeneous equation and a particular solution of the nonhomogeneous equation:

$$\bar{T}(\alpha,t) = A e^{-\alpha^2 t} + \sqrt{\frac{2}{\pi}}\, \frac{T_0}{\alpha}. \tag{9.73}$$

The constant A can be determined from the Fourier transform of the initial condition, Eq. 9.68c, $\bar{T}(\alpha,0) = 0$, which implies

$$A = -\sqrt{\frac{2}{\pi}}\, \frac{T_0}{\alpha}. \tag{9.74}$$

Hence,

$$\bar{T}(\alpha, t) = \sqrt{\frac{2}{\pi}}\, T_0 \left(\frac{1 - e^{-\alpha^2 t}}{\alpha} \right). \tag{9.75}$$

The desired solution $T(x, t)$ is then obtained by taking the inverse Fourier sine transform of this function:

$$T(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{T}(\alpha, t) \sin \alpha x \, d\alpha = \frac{2T_0}{\pi} \int_0^\infty \left(\frac{1 - e^{-\alpha^2 t}}{\alpha} \right) \sin \alpha x \, d\alpha. \tag{9.76}$$

Eq. 9.76 can be viewed as a formal solution of the problem, Eq. 9.68, since the solution has been reduced to evaluating an integral, but this solution can be carried further. Since the first part of the integral in Eq. 9.76 is given by Eq. 9.44, we obtain

$$T(x, t) = T_0 - \frac{2T_0}{\pi} \int_0^\infty \frac{e^{-\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha, \tag{9.77}$$

where

$$\int_0^\infty \frac{e^{-\alpha^2 t}}{\alpha} \sin \alpha x \, d\alpha = \int_0^\infty e^{-\alpha^2 t} \left(\int_0^x \cos \alpha x \, dx \right) d\alpha \tag{9.78}$$

$$= \int_0^x \int_0^\infty e^{-\alpha^2 t} \cos \alpha x \, d\alpha \, dx. \tag{9.79}$$

We now let $I(x)$ denote the inner integral

$$I(x) = \int_0^\infty e^{-\alpha^2 t} \cos \alpha x \, d\alpha, \tag{9.80}$$

in which case

$$\frac{dI}{dx} = -\int_0^\infty \alpha e^{-\alpha^2 t} \sin \alpha x \, d\alpha, \tag{9.81}$$

which can be integrated by parts with the associations

$$u = \sin \alpha x, \quad dv = \alpha e^{-\alpha^2 t} \, d\alpha, \quad v = -\frac{1}{2t} e^{-\alpha^2 t}. \tag{9.82}$$

Then,

$$\frac{dI}{dx} = \frac{1}{2t} \left[\left(e^{-\alpha^2 t} \sin \alpha x \right) \Big|_0^\infty - x \int_0^\infty e^{-\alpha^2 t} \cos \alpha x \, d\alpha \right], \tag{9.83}$$

where the boundary term vanishes, and the integral is $I(x)$. $I(x)$ thus satisfies the ordinary differential equation

$$\frac{dI}{dx} + \frac{x}{2t} I = 0, \tag{9.84}$$

which can be integrated directly to yield

$$I(x) = I_0 e^{-\frac{x^2}{4t}}, \tag{9.85}$$

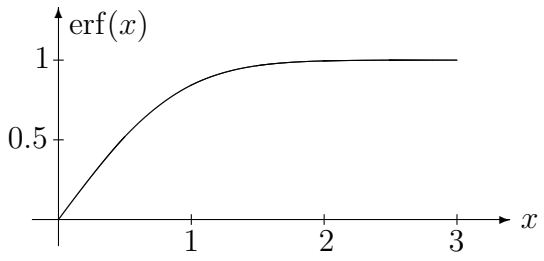


Figure 54: The Error Function.

where

$$I_0 = I(0) = \int_0^\infty e^{-\alpha^2 t}\, d\alpha. \tag{9.86}$$

With the change of variable $w = \alpha\sqrt{t}$, this equation becomes

$$I_0 = \frac{1}{\sqrt{t}} \int_0^\infty e^{-w^2}\, dw = \frac{1}{\sqrt{t}} \cdot \frac{\sqrt{\pi}}{2} \operatorname{erf}(\infty) = \frac{1}{2}\sqrt{\frac{\pi}{t}}, \tag{9.87}$$

where the *error function* (Fig. 54) is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-w^2}\, dw, \tag{9.88}$$

with $\operatorname{erf}(0) = 0$, and $\operatorname{erf}(\infty) = 1$.

Thus, from Eq. 9.85,

$$I(x) = \frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-\frac{x^2}{4t}} \tag{9.89}$$

and, from Eq. 9.77,

$$T(x,t) = T_0 - \frac{2T_0}{\pi} \int_0^x I(x)\, dx = T_0 - \frac{2T_0}{\pi} \cdot \frac{1}{2}\sqrt{\frac{\pi}{t}} \int_0^x e^{-\frac{x^2}{4t}}\, dx. \tag{9.90}$$

With the change of variable $y = x/(2\sqrt{t})$, this equation becomes

$$T(x,t) = T_0 - \frac{2T_0}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{t}}} e^{-y^2}\, dy \tag{9.91}$$

or

$$T(x,t) = T_0 \left[1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right]. \tag{9.92}$$

9.5 Example 2

Consider the following two-dimensional problem on the upper half-space:

$$\begin{aligned} &\nabla^2 T = 0, \quad -\infty < x < \infty, \quad y > 0, \\ &T(x,0) = \begin{cases} 1, & |x| < a, \\ 0, & |x| > a, \end{cases} \\ &\lim_{r\to\infty} T = 0, \end{aligned} \tag{9.93}$$

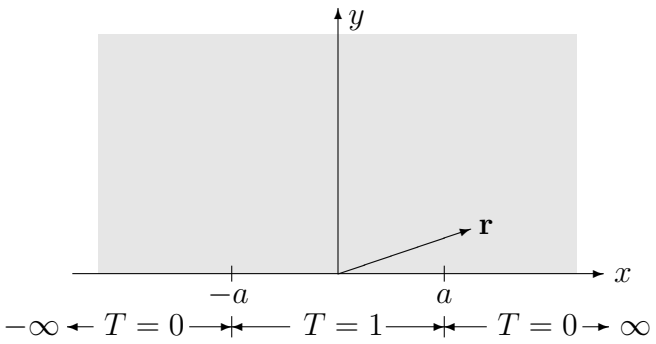


Figure 55: Upper-Half Space With Non-Uniform Dirichlet B.C.

where $r = \sqrt{x^2 + y^2}$ and $T = T(x, y)$ (Fig. 55).

There are three ways to use Fourier transforms to solve this problem:

1. exponential transform in x, since $-\infty < x < \infty$
2. sine transform in y, since $0 < y < \infty$, and the boundary condition at $y = 0$ is on T rather than $\partial T/\partial y$
3. cosine transform in x, since, due to symmetry with respect to the line $x = 0$, an equivalent problem is the quarter-space problem

$$\begin{aligned} &\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad 0 < x < \infty, \quad y > 0, \\ &T(x,0) = \begin{cases} 1, & x < a, \\ 0, & x > a, \end{cases} \\ &\frac{\partial T(0,y)}{\partial x} = 0, \\ &\lim_{r \to \infty} T = 0, \end{aligned} \tag{9.94}$$

where $r = \sqrt{x^2 + y^2}$ and $T = T(x, y)$. That is, the cosine transform in x can be used, since $0 < x < \infty$, and the boundary condition at $x = 0$ is on $\partial T/\partial x$ rather than T.

Consider the third approach. The Fourier cosine transform of the partial differential equation, Eq. 9.94a, is

$$-\sqrt{\frac{2}{\pi}}\,\frac{\partial T(0,y)}{\partial x} - \alpha^2 \bar{T}_c(\alpha, y) + \frac{\partial^2 \bar{T}_c(\alpha, y)}{\partial y^2} = 0 \tag{9.95}$$

where, because of the boundary condition, the first term vanishes. Thus,

$$\frac{\partial^2 \bar{T}_c}{\partial y^2} - \alpha^2 \bar{T}_c = 0, \tag{9.96}$$

which is an ordinary differential equation in y. The general solution of this equation is

$$\bar{T}_c(\alpha, y) = Ae^{\alpha y} + Be^{-\alpha y}, \tag{9.97}$$

where, from the definition of the Fourier cosine transform, $\alpha > 0$. Since $T(x, y)$, and hence $\bar{T}_c(\alpha, y)$, must be finite for large y, $A = 0$. The constant B is evaluated from the boundary condition at $y = 0$:

$$B = \bar{T}_c(\alpha, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty T(x, 0) \cos \alpha x \, dx \tag{9.98}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \frac{\sin \alpha a}{\alpha}. \tag{9.99}$$

Thus,

$$\bar{T}_c(\alpha, y) = \sqrt{\frac{2}{\pi}} \frac{\sin \alpha a}{\alpha} e^{-\alpha y}, \tag{9.100}$$

from which we can use the inverse Fourier cosine transform, Eq. 9.26, to obtain the formal solution

$$T(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sin \alpha a}{\alpha} e^{-\alpha y} \cos \alpha x \, d\alpha. \tag{9.101}$$

9.6 Example 3

We now solve the problem of Example 2 using Method 1, the exponential transform in x, instead of the cosine transform used in the preceding section. The exponential transform of Laplace's equation, Eq. 9.93a, is

$$(-i\alpha)^2 \bar{T} + \frac{\partial^2 \bar{T}}{\partial y^2} = 0 \tag{9.102}$$

or

$$\frac{\partial^2 \bar{T}}{\partial y^2} - \alpha^2 \bar{T} = 0, \tag{9.103}$$

which is the same equation obtained in the preceding section, Eq. 9.96. The general solution of this equation is

$$\bar{T}(\alpha, y) = Ae^{\alpha y} + Be^{-\alpha y}, \tag{9.104}$$

where, from the definition of the exponential transform, $-\infty < \alpha < \infty$.

Finiteness of $T(x, y)$, and hence $\bar{T}(\alpha, y)$, for large y implies that $A = 0$ when $\alpha > 0$, and $B = 0$ when $\alpha < 0$. That is,

$$\bar{T}(\alpha, y) = \begin{cases} Ce^{-\alpha y}, & \alpha > 0, \\ Ce^{\alpha y}, & \alpha < 0, \end{cases} \tag{9.105}$$

or

$$\bar{T}(\alpha, y) = Ce^{-|\alpha| y}. \tag{9.106}$$

The constant C is evaluated using the boundary condition at $y = 0$:

$$C = \bar{T}(\alpha, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty T(x, 0) e^{i\alpha x} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-a}^a e^{i\alpha x} \, dx \tag{9.107}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (\cos \alpha x + i \sin \alpha x) \, dx, \tag{9.108}$$

where the two terms of the integrand are, respectively, even and odd functions of x, so that

$$C = \frac{2}{\sqrt{2\pi}} \int_0^a \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \frac{\sin \alpha a}{\alpha}. \tag{9.109}$$

Thus,

$$\bar{T}(\alpha, y) = \sqrt{\frac{2}{\pi}} \frac{\sin \alpha a}{\alpha} \, e^{-|\alpha| y}, \tag{9.110}$$

on which we can use the inverse Fourier transform, Eq. 9.22, to obtain

$$T(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha} \, e^{-|\alpha| y} \, e^{-i \alpha x} \, d\alpha \tag{9.111}$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha} \, e^{-|\alpha| y} \, (\cos \alpha x - i \sin \alpha x) \, d\alpha, \tag{9.112}$$

where the two terms of the integrand are, respectively, even and odd functions of α. Thus,

$$T(x, y) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \alpha a}{\alpha} \, e^{-\alpha y} \, \cos \alpha x \, d\alpha, \tag{9.113}$$

which is the same result obtained previously in Eq. 9.101. Note that, even though some intermediate results are complex, the final solution is real, as required.

9.7 Example 4: The Infinite String and the d'Alembert Solution

The time-dependent transverse response of an infinitely long string satisfies the one-dimensional wave equation with nonzero initial displacement and velocity specified:

$$\begin{cases} \dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{c^2} \dfrac{\partial^2 y}{\partial t^2}, & -\infty < x < \infty, \quad t > 0, \\ y(x, 0) = f(x), \quad \dfrac{\partial y(x, 0)}{\partial t} = g(x) \\ \lim\limits_{x \to \pm\infty} y(x, t) = 0, \end{cases} \tag{9.114}$$

where $y = y(x, t)$, $f(x)$ is the initial displacement, and $g(x)$ is the initial velocity.

Since $-\infty < x < \infty$, and there is no symmetry to exploit, we will attempt a solution using the exponential Fourier transform in x. That is, $y(x, t)$ will transform to $\bar{y}(\alpha, t)$. The transform of Eq. 9.114a is

$$(-i\alpha)^2 \bar{y} = \frac{1}{c^2} \frac{\partial^2 \bar{y}}{\partial t^2} \tag{9.115}$$

or

$$\frac{\partial^2 \bar{y}}{\partial t^2} + (\alpha c)^2 \bar{y} = 0. \tag{9.116}$$

This is an ordinary differential equation in time with the general solution

$$\bar{y}(\alpha, t) = A \sin(\alpha c t) + B \cos(\alpha c t). \tag{9.117}$$

The constants A and B are evaluated using the initial conditions, Eq. 9.114b. From Eq. 9.117,

$$B = \bar{y}(\alpha, 0) = \bar{f}(\alpha), \tag{9.118}$$

where $\bar{f}(\alpha)$ is the Fourier transform of the initial displacement $f(x)$. Also, since

$$\frac{\partial \bar{y}}{\partial t} = \alpha c A \cos(\alpha c t) - \alpha c B \sin(\alpha c t), \tag{9.119}$$

it follows that

$$\alpha c A = \frac{\partial \bar{y}(\alpha, 0)}{\partial t} = \bar{g}(\alpha), \tag{9.120}$$

where $\bar{g}(\alpha)$ is the Fourier transform of the initial velocity $g(x)$. Thus, from Eq. 9.117,

$$\bar{y}(\alpha, t) = \frac{\bar{g}(\alpha)}{\alpha c} \sin(\alpha c t) + \bar{f}(\alpha) \cos(\alpha c t). \tag{9.121}$$

Both $\bar{f}(\alpha)$ and $\bar{g}(\alpha)$ are assumed to exist. The desired solution is then obtained formally from the inverse transform

$$y(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{\bar{g}(\alpha)}{\alpha c} \sin(\alpha c t) + \bar{f}(\alpha) \cos(\alpha c t) \right] e^{-i\alpha x} \, d\alpha. \tag{9.122}$$

Like the problem of Example 1, this solution can be simplified further. Eq. 9.122 can be written

$$\begin{aligned}
y(x,t) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\bar{g}(\alpha) \left(\int_0^t \cos \alpha c t' \, dt' \right) + \bar{f}(\alpha) \cos(\alpha c t) \right] e^{-i\alpha x} \, d\alpha \\
=& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha) \left(\frac{e^{i\alpha c t} + e^{-i\alpha c t}}{2} \right) e^{-i\alpha x} \, d\alpha \\
& + \int_0^t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{g}(\alpha) \left(\frac{e^{i\alpha c t'} + e^{-i\alpha c t'}}{2} \right) e^{-i\alpha x} \, d\alpha \, dt' \\
=& \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(\alpha) \left[e^{-i\alpha(x-ct)} + e^{-i\alpha(x+ct)} \right] d\alpha \\
& + \frac{1}{2} \int_0^t \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{g}(\alpha) \left[e^{-i\alpha(x-ct')} + e^{-i\alpha(x+ct')} \right] d\alpha \, dt' \\
=& \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2} \int_0^t g(x-ct') \, dt' + \frac{1}{2} \int_0^t g(x+ct') \, dt'.
\end{aligned} \tag{9.123}$$

We now make the changes of variable $\tau = x - ct'$ in the first integral and $\tau = x + ct'$ in the second integral to obtain

$$y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] - \frac{1}{2c} \int_x^{x-ct} g(\tau) \, d\tau + \frac{1}{2c} \int_x^{x+ct} g(\tau) \, d\tau \tag{9.124}$$

or

$$y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\tau) \, d\tau, \tag{9.125}$$

which is the d'Alembert solution given earlier.

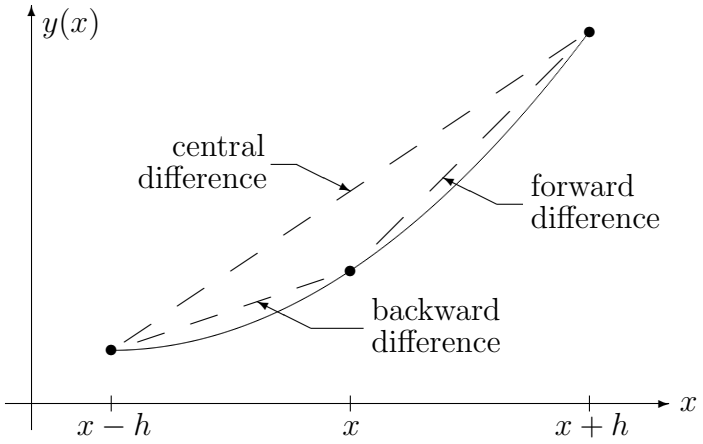


Figure 56: Finite Difference Approximations to Derivatives.

10 Finite Difference Solutions

10.1 Derivatives

Consider a function $y(x)$ for which we want to approximate the derivative $y'(x)$ at some point x. If we discretize the x-axis with uniform spacing h, we could approximate the derivative using the *forward difference* formula

$$y'(x) \approx \frac{y(x+h) - y(x)}{h}, \tag{10.1}$$

which is the slope of the line to the right of x (Fig. 56). We could also approximate the derivative using the *backward difference* formula

$$y'(x) \approx \frac{y(x) - y(x-h)}{h}, \tag{10.2}$$

which is the slope of the line to the left of x. Since, in general, there is no basis for choosing one of these approximations over the other, an intuitively more appealing approximation results from the average of these formulas:

$$y'(x) \approx \frac{1}{2}\left[\frac{y(x+h) - y(x)}{h} + \frac{y(x) - y(x-h)}{h}\right] \tag{10.3}$$

or

$$y'(x) \approx \frac{y(x+h) - y(x-h)}{2h}. \tag{10.4}$$

This formula, which is more accurate than either the forward or backward difference formulas, is the *central finite difference approximation* to the derivative.

The central difference formula for second derivatives can be derived using Taylor series expansions:

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2}y''(x) + \frac{h^3}{6}y'''(x) + O(h^4), \tag{10.5}$$

where we use the "big O" notation $O(h)$ to represent terms of order h as $h \to 0$. Similarly, by replacing h by $-h$,

$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2}y''(x) - \frac{h^3}{6}y'''(x) + O(h^4). \tag{10.6}$$

The addition of these two equations yields

$$y(x+h) + y(x-h) = 2y(x) + h^2 y''(x) + O(h^4) \tag{10.7}$$

or

$$y''(x) = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + O(h^2), \tag{10.8}$$

which, because of the error term, shows that the formula is second-order accurate.

10.2 Heat Equation

Consider the boundary-initial value problem (BIVP)

$$\begin{cases} u_{xx} = \dfrac{1}{c}\, u_t, \quad u = u(x,t), \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0 \quad \text{(boundary conditions)} \\ u(x,0) = f(x) \quad \text{(initial condition)}, \end{cases} \tag{10.9}$$

where c is a constant. This problem represents transient heat conduction in a rod with the ends held at zero temperature and an initial temperature profile $f(x)$.

To solve this problem numerically, we discretize x and t such that

$$\begin{aligned} x_i &= ih, \quad i = 0, 1, 2, \ldots \\ t_j &= jk, \quad j = 0, 1, 2, \ldots . \end{aligned} \tag{10.10}$$

That is, the spatial mesh spacing is h, and the time mesh spacing is k.

Let $u_{i,j}$ be the numerical approximation to $u(x_i, t_j)$. If we approximate u_t with forward finite differences and u_{xx} with central finite differences, the finite difference approximation to the PDE is then

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = \frac{u_{i,j+1} - u_{i,j}}{ck}. \tag{10.11}$$

Define the parameter r as

$$r = \frac{ck}{h^2} = \frac{c\,\Delta t}{(\Delta x)^2}, \tag{10.12}$$

in which case Eq. 10.11 becomes

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}. \tag{10.13}$$

The domain of the problem and the mesh are illustrated in Fig. 57. Eq. 10.13 is a recursive relationship giving u in a given row (time) in terms of three consecutive values of u in the row below (one time step earlier). This equation is referred to as an *explicit* formula

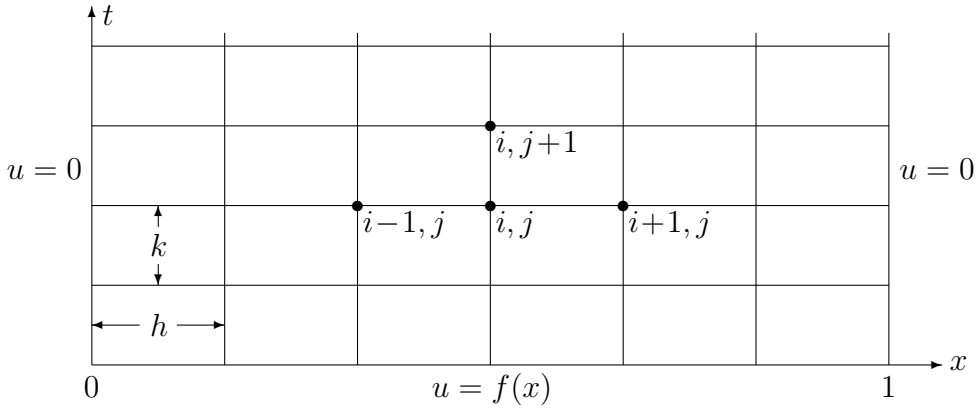


Figure 57: Mesh for 1-D Heat Equation.

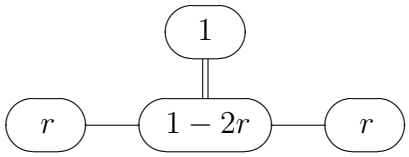


Figure 58: Heat Equation Stencil for Explicit Finite Difference Algorithm.

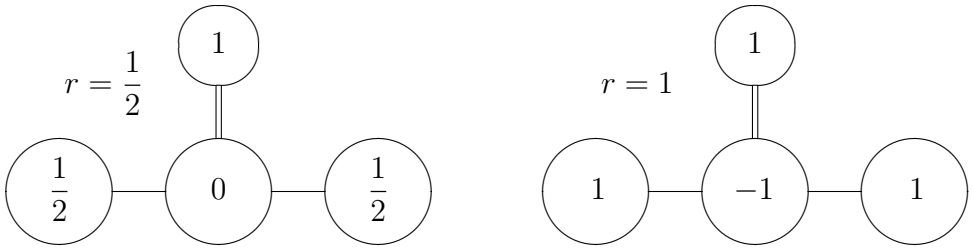


Figure 59: Heat Equation Stencils for $r = 1/2$ and $r = 1$.

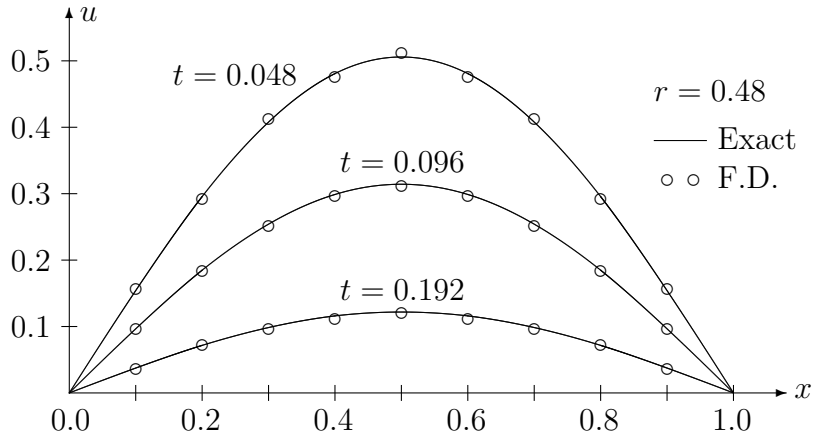


Figure 60: Explicit Finite Difference Solution With $r = 0.48$ (Stable).

since one unknown value can be found directly in terms of several other known values. The recursive relationship can also be sketched with the stencil shown in Fig. 58.

Notice that, if $r = 1/2$, the solution at the new point is *independent* of the closest point, as illustrated in Fig. 59. For $r > 1/2$ (e.g., $r = 1$), the new point depends *negatively* on the closest point (Fig. 59), which is counter-intuitive. It can be shown that, for a stable solution, $0 < r \leq 1/2$. An unstable solution is one for which small errors grow rather than decay as the solution evolves.

The instability which occurs for $r > 1/2$ can be illustrated with the following example. Consider the boundary-initial value problem (in nondimensional form)

$$\begin{aligned} &u_{xx} = u_t, \quad 0 < x < 1, \quad u = u(x,t) \\ &u(0,t) = u(1,t) = 0 \\ &u(x,0) = f(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2 \\ 2(1-x), & 1/2 \leq x \leq 1. \end{cases} \end{aligned} \tag{10.14}$$

The physical problem is to compute the temperature history $u(x,t)$ for a bar with a prescribed initial temperature distribution $f(x)$, no internal heat sources, and zero temperature prescribed at both ends. We solve this problem using the explicit finite difference algorithm with $h = \Delta x = 0.1$ and $k = \Delta t = rh^2 = r(\Delta x)^2$ for two different values of r: $r = 0.48$ and $r = 0.52$. The two numerical solutions (Figs. 60 and 61) are compared with the analytic solution

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8}{(n\pi)^2} \sin\frac{n\pi}{2} (\sin n\pi x)\, e^{-(n\pi)^2 t}, \tag{10.15}$$

which can be obtained by the technique of separation of variables. The instability for $r > 1/2$ can be clearly seen in Fig. 61. Thus, a disadvantage of this explicit method is that a small time step Δt must be used to maintain stability. This disadvantage can be removed with implicit methods such as the Crank-Nicolson algorithm.

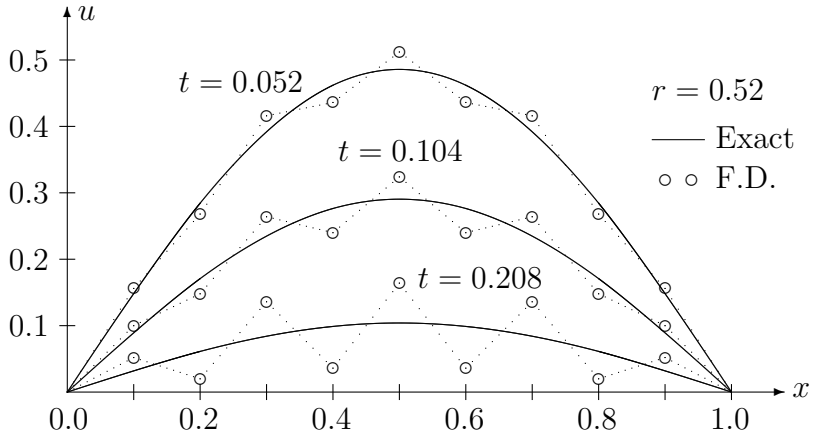


Figure 61: Explicit Finite Difference Solution With $r = 0.52$ (Unstable).

10.3 Wave Equation

Consider the boundary-initial value problem (BIVP)

$$\begin{cases} u_{xx} = \dfrac{1}{c^2}\, u_{tt}, \quad u = u(x,t), \quad 0 < x < a, \quad t > 0 \\ u(0,t) = u(a,t) = 0 \quad \text{(boundary conditions)} \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{(initial conditions)}. \end{cases} \tag{10.16}$$

This problem represents the transient (time-dependent) vibrations of a string fixed at the two ends with both initial displacement $f(x)$ and initial velocity $g(x)$ specified.

A central finite difference approximation to the PDE, Eq. 10.16, yields

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{c^2 k^2}. \tag{10.17}$$

We define the parameter

$$r = \frac{ck}{h} = \frac{c\,\Delta t}{\Delta x}, \tag{10.18}$$

and solve for $u_{i,j+1}$:

$$u_{i,j+1} = r^2 u_{i-1,j} + 2(1 - r^2)u_{i,j} + r^2 u_{i+1,j} - u_{i,j-1}. \tag{10.19}$$

Fig. 62 shows the mesh points involved in this recursive scheme. If we know the solution for all time values up to the jth time step, we can compute the solution $u_{i,j+1}$ at Step $j + 1$ in terms of known quantities. Thus, this algorithm is an explicit algorithm. The corresponding stencil is shown in Fig. 63.

It can be shown that this finite difference algorithm is stable if $r \leq 1$ and unstable if $r > 1$. This stability condition is know as the Courant, Friedrichs, and Lewy (CFL) condition or simply the Courant condition. It can be shown that Δt should be selected as large as possible without exceeding the stability limit of $r = 1$.

An intuitive rationale behind the stability requirement $r \leq 1$ can also be made using Fig. 64. If $r > 1$, the numerical domain of dependence (NDD) would be smaller than the

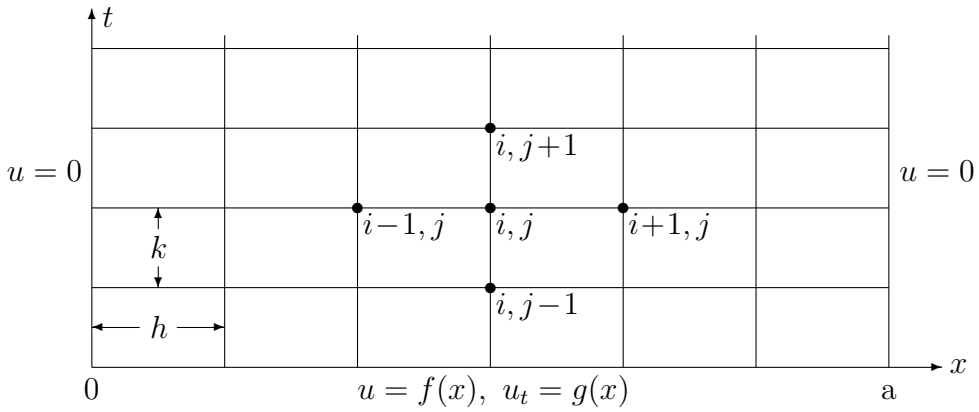


Figure 62: Mesh for Explicit Solution of Wave Equation.

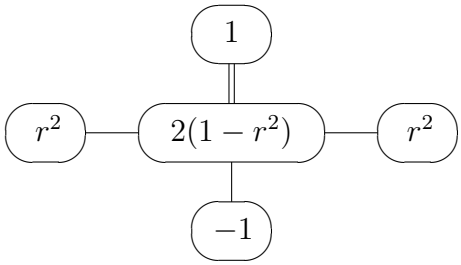


Figure 63: Stencil for Explicit Solution of Wave Equation.

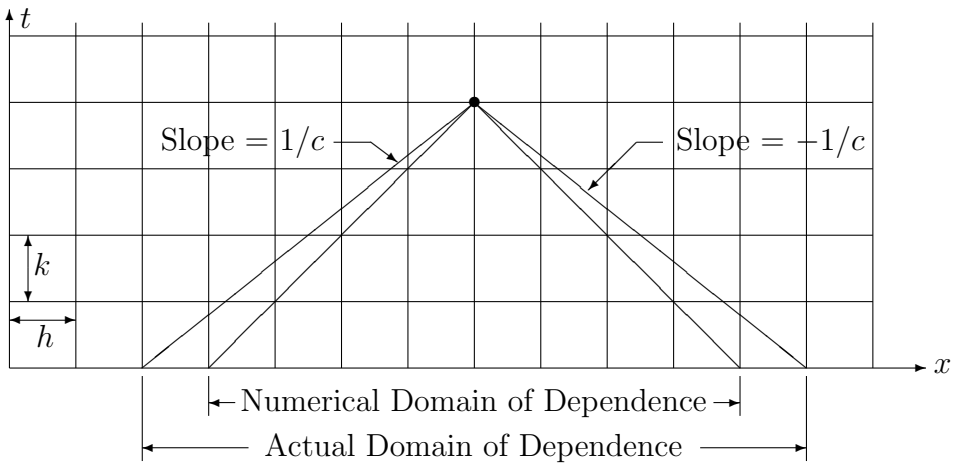


Figure 64: Domains of Dependence for $r > 1$.

actual domain of dependence (ADD) for the PDE, since NDD spreads by one mesh point at each level for earlier time. However, if NDD < ADD, the numerical solution would be independent of data outside NDD but inside ADD. That is, the numerical solution would ignore necessary information. Thus, to insure a stable solution, $r \leq 1$.

We note that the explicit finite difference scheme just described for the wave equation requires the numerical solution at two consecutive time steps to step forward to the next time. Thus, at $t = 0$, a special procedure is needed to advance from $j = 0$ to $j = 1$. To compute the solution at the end of the first time step, let $j = 0$ in Eq. 10.19 to obtain

$$u_{i,1} = r^2 u_{i-1,0} + 2(1 - r^2)u_{i,0} + r^2 u_{i+1,0} - u_{i,-1}, \tag{10.20}$$

where the right-hand side is known (from the initial condition) except for $u_{i,-1}$. However, we can write a central difference approximation to the first time derivative at $t = 0$:

$$\frac{u_{i,1} - u_{i,-1}}{2k} = g_i \tag{10.21}$$

or

$$u_{i,-1} = u_{i,1} - 2kg_i, \tag{10.22}$$

where g_i is the initial velocity $g(x)$ evaluated at the ith point, i.e., $g_i = g(x_i)$. If we substitute this last result into Eq. 10.20 (to eliminate $u_{i,-1}$), we obtain

$$u_{i,1} = \frac{1}{2}r^2 u_{i-1,0} + (1 - r^2)u_{i,0} + \frac{1}{2}r^2 u_{i+1,0} + kg_i. \tag{10.23}$$

This is the difference equation used for the first row. Thus, to implement the explicit finite difference algorithm, we use Eq. 10.23 for the first time step and Eq. 10.19 for all subsequent time steps.

10.4 Laplace's Equation

Consider Laplace's equation on the two-dimensional rectangular domain shown in Fig. 65:

$$\begin{cases} \nabla^2 u(x, y) = 0 \quad (0 < x < a,\ 0 < y < b), \\ u(0, y) = g_1(y), \quad u(a, y) = g_2(y), \quad u(x, 0) = f_1(x), \quad u(x, b) = f_2(x). \end{cases} \tag{10.24}$$

This problem corresponds physically to two-dimensional steady-state heat conduction over a rectangular plate for which temperature is specified on the boundary.

We attempt an approximate solution by introducing a uniform rectangular grid over the domain, and let the point (i, j) denote the point having the ith value of x and the jth value of y (Fig. 66). Then, using central finite difference approximations to the second derivatives (Fig. 67), the finite difference approximation to Laplace's equation thus becomes

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} = 0 \tag{10.25}$$

or

$$4u_{i,j} - (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}) = 0. \tag{10.26}$$

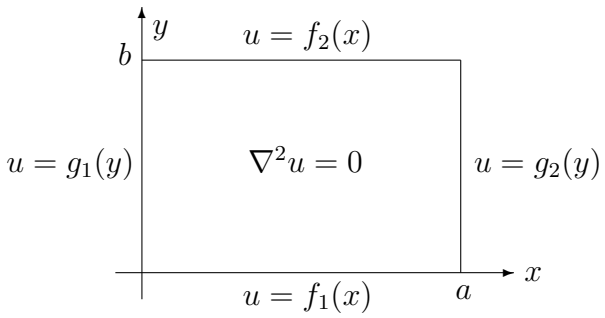


Figure 65: Laplace's Equation on Rectangular Domain.

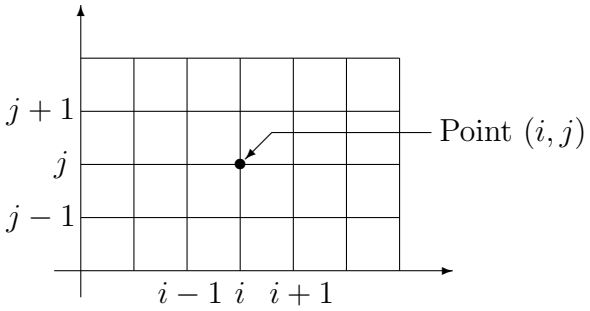


Figure 66: Finite Difference Grid on Rectangular Domain.

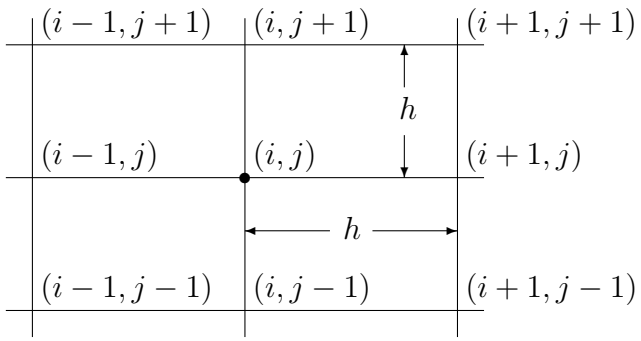


Figure 67: The Neighborhood of Point (i, j).

4	8	12	16	20
3	7	11	15	19
2	6	10	14	18
1	5	9	13	17

Figure 68: 20-Point Finite Difference Mesh.

That is, for Laplace's equation with the same uniform mesh in each direction, the solution at a typical point (i, j) is the *average* of the four neighboring points.

For example, consider the mesh shown in Fig. 68. Although there are 20 mesh points, 14 are on the boundary, where the solution is known. Thus, the resulting numerical problem has only six degrees of freedom (unknown variables). The application of Eq. 10.26 to each of the six interior points yields

$$\begin{cases} 4u_6 - u_7 - u_{10} = u_2 + u_5 \\ -u_6 + 4u_7 - u_{11} = u_3 + u_8 \\ -u_6 + 4u_{10} - u_{11} - u_{14} = u_9 \\ -u_7 - u_{10} + 4u_{11} - u_{15} = u_{12} \\ -u_{10} + 4u_{14} - u_{15} = u_{13} + u_{18} \\ -u_{11} - u_{14} + 4u_{15} = u_{16} + u_{19}, \end{cases} \tag{10.27}$$

where all known quantities have been placed on the right-hand side. This linear system of six equations in six unknowns can be solved with standard equation solvers. Because the central difference operator is a 5-point operator, systems of equations of this type would have at most five nonzero terms in each equation, regardless of how large the mesh is. Thus, for large meshes, the system of equations is sparsely populated, so that sparse matrix solution techniques would be applicable.

Systems of this type can also be solved using an iterative procedure known as *relaxation*, which uses the following general algorithm:

1. Initialize the boundary points to their prescribed values, and initialize the interior points to zero or some other convenient value (e.g., the average of the boundary values).

2. Loop systematically through the interior mesh points, setting each interior point to the average of its four neighbors.

3. Continue this process until the solution converges to the desired accuracy.

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Index