Viscosity Effects on the Propagation of Acoustic Transients

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The propagation of an impulsive excitation applied at the origin of a lossy viscous medium is studied by operational techniques as the excitation advances through the medium. The solution of the governing partial differential equation (PDE) for such transient propagation problems has been elusive. Such solution is found as an infinite sum of properly weighted successive integrals of the complementary error function, and it is quantitatively examined here using a one-dimensional model in space and time. As expected, as the transient advances through space, its amplitude decreases, and its width broadens. Such is the damping effect of viscosity that one would anticipate from elementary considerations in related disciplines such as electrodynamics. Such is also the smoothing-out effect of dispersion. We also obtain an approximate solution of the present boundary-initial value problem based on the method of steepest descents. This approximation agrees with the first term of the complete analytic solution given here. The pertinent dispersion relation associated with the governing parabolic PDE is shown to impose a restrictive condition on the allowable values of the propagation speed and the kinematic viscosity coefficient, thus assuring that propagation with attenuation does take place. Various numerical results illustrate and quantitatively describe the propagation of the transient pulse in several nondimensional graphs. [DOI: 10.1115/1.1419203]

Introduction

The propagation of a sound pulse through a viscous fluid is an important problem that has been often revisited since the governing equation for the propagation was established by Stokes [1] in 1845. Various, more or less generalized, formulations have appeared in textbooks [2–5] as well as in papers [6–8]. The governing equation is a third-order partial differential equation of parabolic type. In one space dimension, the boundary/initial value problem [BIVP] is

$$\begin{cases} \dfrac{\nu}{c^2}\dfrac{\partial^3 u}{\partial t\partial x^2}+\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} \quad (x>0,t>0), & (a) \\ u(x,0)=0, \qquad \dot{u}(x,0)=0, & (b) \\ u(0,t)=u_0\delta(t), \qquad u(\infty,t)<\infty, & \end{cases} \tag{1}$$

where $u(x,t)$ is the fluid velocity, c is the sound speed, ν is the kinematic viscosity responsible for the attenuation, $\dot{u}$ is the acceleration, and $\delta(t)$ is the Dirac delta function.

Works dealing with some aspects of this problem are discussed in some textbooks. For example, in Lamb [[2], p. 647], Malecki [[3], p. 260], and Officer [[4], pp. 249–257], a harmonic wave solution is assumed and substituted into the PDE. This substitution yields classical, linearly attenuated harmonic waves at retarded time. These waves are not solutions of transient boundary and initial value problems such as Eq. (1) since no account was taken of the boundary or initial conditions. In Morse and Ingard [[5], p. 282], the interest was in the formulation of problems in which viscosity and heat conduction were jointly present, starting from the basic equations of fluid mechanics. Their formulation leads to a system of three coupled PDEs for three pertinent state variables, and when heat conduction becomes negligible, the system of equations reduced to Eq. (1) for the fluid velocity. However, no solution was offered.

In the more detailed work by Blackstock [6], a BIVP consisting of Eq. (1) together with a sinusoidal boundary condition, was posed. Blackstock used the "smallness approximation" to reduce the PDE in our Eq. (1) to his Eq. (14), which is the heat equation. The heat equation was then solved for the sinusoidal wave train that was initiated at the origin and later decayed to zero at large distances.

A purely numerical treatment of such transient problems has also been given by Ludwig and Levin [7]. The "smallness approximation" was also used by Szabo [8] in 1968, who stated that the analytic solution of Eq. (1) was "not easy to obtain." He then proceeded to design an electrical analog system to simulate system responses under such restriction.

Finally, Lagerstrom, Cole, and Trilling [9], in a report pointed out to us by a reviewer, obtained an asymptotic expansion for the inversion integral associated with the BIVP, Eq. (1), for the case of large values of time *t*. We will return to this asymptotic approximation in our final discussion, since we can show that the approximation also emerges from the present complete solution of the BIVP.

In the present problem, there is propagation in only one direction into the infinite viscous medium. The radiation condition, implicit in this work, prevents waves from returning in the "inward direction." Of particular interest is the solution when the boundary condition is an impulsive excitation at the origin. If such solution were obtained, then it could be used, by means of the convolution theorem, to find solutions for any other excitation. Since the Laplace transform used here is one-sided, the convolution theorem is ideally suited for this purpose. This basic solution will reveal the damping effects of viscosity in a quantitative way. Qualitative analyses in the literature [[10], pp. 212–215] have exhibited the anticipated behavior in which the pulse amplitude decreases, and its width broadens, as it propagates.

We note that parabolic PDEs similar to this often have precursor solutions with infinite speed of propagation, so that disturbances are felt immediately at all points in the fluid. This property allows for the smoothing-out effect of dispersion within the context of continuum mechanics. This issue has been covered in detail by Weymann in 1967 [11] and will not be considered here. In

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the following sections, we derive the analytic solution obtained by operational methods as well as an approximation to the analytical solution based on the method of steepest descents, which will identify the strongest contribution within the complete analytic solution to the value of the pertinent contour integral.

Theory

A Laplace transform pair is here defined as

$$\begin{cases} \hat{f}(s) = \int_0^\infty f(t) e^{-st} dt, \\ f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{f}(s) e^{st} ds, \end{cases} \tag{2}$$

where

$$\mathcal{L}\{f'(t)\} = s\hat{f}(s) - f(0+). \tag{3}$$

We take the Laplace transform of Eq. 1 with respect to time t to obtain

$$\begin{cases} (1+\tau s)\hat{u}_{xx}(x,s) - \frac{s^2}{c^2}\hat{u}(x,s) = 0, \\ \hat{u}(0,s) = u_0, \end{cases} \tag{4}$$

where $\hat{u}(x,s)$ is the transform of $u(x,t)$, subscripts denote partial differentiation, $\tau = \nu/c^2$, and, because of the initial condition, we take $u_{xx}(x,0) = 0$. Since ν has the dimensions $[L^2/T]$, τ has the dimensions of time. Because of the transform, u_0 now takes the dimensions of length. The solution of this system is

$$\hat{u}(x,s) = u_0 \exp\left(-\frac{sx}{c\sqrt{1+\tau s}} \right). \tag{5}$$

We now take the inverse Laplace transform of this last equation to obtain

$$u(x,t) = u_0 \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{-(sx/c\sqrt{1+\tau s})} e^{st} ds, \tag{6}$$

which has an essential singularity at $s = -1/\tau$. With the change of variable

$$s = p - 1/\tau, \tag{7}$$

Equation (6) becomes

$$u(x,t) = u_0 \frac{e^{-t/\tau}}{2\pi i} \int_{\sigma + 1/\tau - i\infty}^{\sigma + 1/\tau + i\infty} e^{x/c\tau(1/\sqrt{\tau p} - \sqrt{\tau p})} e^{pt} dp. \tag{8}$$

We now define the nondimensional variables

$$\bar{x} = \frac{x}{c\tau}, \quad \bar{t} = \frac{t}{\tau}, \tag{9}$$

in which case Eq. (8) becomes

$$u(x,t) = u_0 \frac{e^{-\bar{t}}}{2\pi i} \int_{\sigma + 1/\tau - i\infty}^{\sigma + 1/\tau + i\infty} e^{\bar{x}(1/\sqrt{\tau p} - \sqrt{\tau p})} e^{pt} dp \tag{10}$$

$$= u_0 \frac{e^{-\bar{t}}}{2\pi i} \int_{\sigma + 1/\tau - i\infty}^{\sigma + 1/\tau + i\infty} F_1(p) F_2(p) e^{pt} dp, \tag{11}$$

where

$$F_1(p) = e^{-\bar{x}\sqrt{\tau p}}, \quad F_2(p) = e^{\bar{x}/\sqrt{\tau p}}. \tag{12}$$

If we expand $F_2(p)$ in a Maclaurin series, the product of these two functions can be written

$$F(p) = F_1(p) F_2(p) \tag{13}$$

$$= e^{-\bar{x}\sqrt{\tau p}} \left[1 + \frac{\bar{x}}{\sqrt{\tau}\sqrt{p}} + \frac{\bar{x}^2}{2!\,\tau p} + \dots + \frac{\bar{x}^n}{n!\,\tau^{n/2} p^{n/2}} + \dots \right] \tag{14}$$

$$= e^{-\bar{x}\sqrt{\tau p}} + \frac{\bar{x}}{\sqrt{\tau}} \cdot \frac{e^{-\bar{x}\sqrt{\tau p}}}{\sqrt{p}} + \frac{\bar{x}^2}{2!\,\tau} \cdot \frac{e^{-\bar{x}\sqrt{\tau p}}}{p} + \dots + \frac{\bar{x}^n}{n!\,\tau^{n/2}} \cdot \frac{e^{-\bar{x}\sqrt{\tau p}}}{p^{n/2}} + \dots, \tag{15}$$

which can be inverted term-by-term.

We note that the first term in Eq. (15) is $F_1(p)$, whose inverse $f_1(t)$ is tabulated [12,13]; hence,

$$f_1(t) = \frac{\bar{x}}{2\tau \bar{t}\sqrt{\pi \bar{t}}} e^{-(\bar{x}^2/4\bar{t})}. \tag{16}$$

The inverse of the second term in Eq. (15) is also tabulated [12]:

$$\mathcal{L}^{-1}\left\{ \frac{\bar{x}}{\sqrt{\tau}} \cdot \frac{e^{-\bar{x}\sqrt{\tau p}}}{\sqrt{p}} \right\} = \frac{\bar{x}}{\tau\sqrt{\pi \bar{t}}} e^{-\bar{x}^2/(4\bar{t})}. \tag{17}$$

Third and subsequent terms in Eq. (15) can be inverted using the known formula [[12], Section 7.2]

$$\mathcal{L}^{-1}\left\{ \frac{e^{-\beta\sqrt{p}}}{p^{n/2}} \right\} = (4t)^{n/2-1} \mathcal{I}^{\,n-2} \operatorname{erfc}\left(\frac{\beta}{2\sqrt{t}} \right), \tag{18}$$

where $\beta = \bar{x}\sqrt{\tau}$, and $\mathcal{I}^{\,m}\operatorname{erfc}(z)$ is a symbolic operator that denotes the mth successive integrals of the complementary error function. (Note that we use the symbol $\mathcal{I}$ rather than the symbol i used by Abramowitz and Stegun for this operator to avoid confusion with the imaginary unit i also used in this paper.) We recall that

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt \tag{19}$$

and the recursion relation [14]

$$\mathcal{I}^{\,n}\operatorname{erfc}(z) = \int_z^\infty \mathcal{I}^{\,n-1}\operatorname{erfc}(t)\,dt, \quad (n = 0,1,2,\dots). \tag{20}$$

Hence the inverse of the nth term in Eq. (15) is, in nondimensional form,

$$\frac{\bar{x}^n}{n!\,\tau^{n/2}} \mathcal{L}^{-1}\left\{ \frac{-e^{-\bar{x}\sqrt{\tau p}}}{p^{n/2}} \right\} = \frac{\bar{x}^n}{\tau n!} (4\bar{t})^{n/2-1} \mathcal{I}^{\,n-2} \operatorname{erfc}\left(\frac{\bar{x}}{2\sqrt{\bar{t}}} \right). \tag{21}$$

The complete inversion is thus obtained by combining Eqs. (16) and (17) with the sum of all $n \geq 2$ terms in Eq. (21):

$$u(\bar{x},\bar{t}) = u_0 \frac{e^{-\bar{t}}}{\tau} \left[\frac{\bar{x}}{\sqrt{\pi \bar{t}}} \left(\frac{1}{2\bar{t}} + 1 \right) e^{-\bar{x}^2/(4\bar{t})} + \sum_{n=2}^\infty \frac{\bar{x}^n}{n!} (4\bar{t})^{n/2-1} \mathcal{I}^{\,n-2} \operatorname{erfc}\left(\frac{\bar{x}}{2\sqrt{\bar{t}}} \right) \right], \tag{22}$$

or, in nondimensional form,

$$\bar{u}(\bar{x},\bar{t}) = e^{-\bar{t}} \left[\frac{1}{\sqrt{\pi}} (1 + 2\bar{t}) \frac{\bar{x}}{2\sqrt{\bar{t}}} e^{-\bar{x}^2/(4\bar{t})} + \sum_{n=2}^\infty \frac{2^{n-2}}{\Gamma(n+1)} \bar{x}^n \bar{t}^{n/2} \mathcal{I}^{n-2} \operatorname{erfc}\left(\frac{\bar{x}}{2\sqrt{\bar{t}}} \right) \right]. \tag{23}$$

The power series expansion of $\mathcal{I}^{\,m}\operatorname{erfc}(z)$ is given by [[12], Section 7.2]

$$\mathcal{I}^m \operatorname{erfc}(z)=\frac{1}{2^m}\sum_{k=0}^{\infty}\frac{(-2z)^k}{\Gamma(k+1)\Gamma\left(1+\frac{m-k}{2}\right)},\tag{24}$$

which can be substituted into Eq. (23) to obtain

$$\overline{u}(\overline{x},\overline{t})=e^{-\overline{t}}\left[\frac{1}{\sqrt{\pi}}(1+2\overline{t})\frac{\overline{x}}{2\sqrt{\overline{t}}}e^{-\overline{x}^2/(4\overline{t})}+\sum_{n=2}^{\infty}\frac{\overline{x}^n}{\Gamma(n+1)}(\overline{t})^{n/2}\sum_{k=0}^{\infty}\frac{(-1)^k(\overline{x}/\sqrt{\overline{t}})^k}{\Gamma(k+1)\Gamma\left(\frac{n-k}{2}\right)}\right]\tag{25}$$

The triple factorial growth in the denominator of the double sum is an indication of the negligible value of its contribution to the solution. In terms of the original, dimensional variables x and t, this result is

$$u(x,t)=u_0e^{-t/\tau}\frac{x}{2c\sqrt{\pi\tau t}}\left(\frac{1}{t}+\frac{2}{\tau}\right)e^{-x^2/(4c^2\tau t)}+u_0\frac{e^{-t/\tau}}{t}\sum_{n=2}^{\infty}\frac{1}{\Gamma(n+1)}\left(\frac{t}{\tau}\right)^{n/2}\left(\frac{x}{c\sqrt{\tau t}}\right)^n\times\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma\left(\frac{n-k}{2}\right)}\left(\frac{x}{c\sqrt{\tau t}}\right)^k.\tag{26}$$

The factor $e^{-t/\tau}$ comes from the shift in Eq. (7) when changing the variable from s to p. Here it is more convenient to deal with and plot the nondimensional Eq. (25), since a single set of curves will quantitatively describe results for all dimensional values of x and t.

We note that, for $n=m+2$, the series in Eq. (23) can be expressed as

$$\sum_{m=0}^{\infty}(2\overline{x}\overline{t})^m\frac{\overline{x}^2}{\Gamma(m+3)}\mathcal{I}^m \operatorname{erfc}(z),$$

where $z=\overline{x}/(2\sqrt{\overline{t}})$. A convenient way to evaluate the series is by means of the recursion relation [12,14]

$$\mathcal{I}^m \operatorname{erfc}(z)=\frac{1}{2m}\mathcal{I}^{m-2}\operatorname{erfc}(z)-\frac{z}{m}\mathcal{I}^{m-1}\operatorname{erfc}(z).\tag{27}$$

All $m\geqslant 1$ terms can be expressed in terms of the first two by the above relation, where the first two are

$$\mathcal{I}^{-1}\operatorname{erfc}(z)=\frac{2}{\sqrt{\pi}}e^{-z^2},\quad \mathcal{I}^0\operatorname{erfc}(z)=\operatorname{erfc}(z).\tag{28}$$

An Approximation

Another way to proceed with the inversion of Eq. (6) is to use the change of variable from s to p given by

$$1+\tau s=p^2,\tag{29}$$

which yields

$$u(x,t)=u_0\frac{1}{\pi i\tau}\int_L \exp\left[\frac{1}{\tau}(p^2-1)\left(t-\frac{x}{cp}\right)\right]p\,dp.\tag{30}$$

In this integral, L is a vertical path to the right of the origin O, and there are an essential singularity at $p=0$ and a branch point at $s=-1/\tau$.

Let $f(p)$ denote the function in the exponent of Eq. (30),

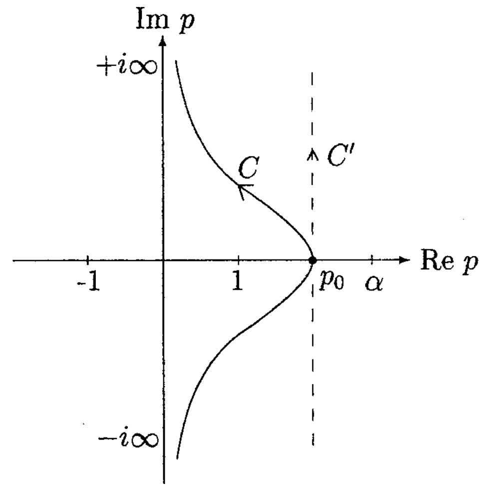

Fig. 1 The p-plane with the locations of p_0, α, and the path C of steepest descents

$$f(p)=\frac{1}{\tau}(p^2-1)\left(t-\frac{x}{cp}\right)=\frac{t}{\tau}(p^2-1)\left(1-\frac{\alpha}{p}\right),\tag{31}$$

where $\alpha=x/(ct)$, from which it follows that

$$\frac{\tau}{t}f'(p)=2p-\alpha-\frac{\alpha}{p^2},\tag{32}$$

$$\frac{\tau}{t}f''(p)=2\left(1+\frac{\alpha}{p^3}\right).\tag{33}$$

The roots of $f'(p)$ are the solutions of

$$2p^3-\alpha p^2-\alpha=0,\tag{34}$$

which has no negative real roots. We note that

$$f'(1)=2-2\alpha,\quad f'(\alpha)=\alpha(\alpha^2-1),\quad f'(\infty)=\infty.\tag{35}$$

There are various cases of interest. Assume $\alpha>1$, in which case the waveform that started at $t=0$ has not yet reached the spatial location x, since $x>ct$. In that case, from Eq. (35),

$$f'(1)<0,\quad f'(\alpha)>0,\quad f'(\infty)>0,\tag{36}$$

which, because of the sign change, implies a root between $p=1$ and $p=\alpha$. We denote that root p_0 (Fig. 1). Similarly, if $\alpha<1$, there is also a root between $p=1$ and $p=\alpha$, except that, in this case, $p=\alpha$ and $p=1$ would exchange places in Fig. 1. In both cases, $f''(p_0)>0$. Thus the path C of steepest descents is parallel to the imaginary axis. This path passes through p_0 and is asymptotic to the imaginary axis for $p\to\pm\infty$. The portion of the path that contributes most to the value of the integral is that near the saddle point p_0. The other roots of $f'(p)=0$ are complex with negative real parts.

We now write the complex number p in terms of its real and imaginary parts,

$$p=p_0+i\sigma,\tag{37}$$

so that Eq. (30) can be written

$$u(x,t)=u_0\frac{1}{\pi i\tau}\int_C \exp\left\{\frac{t}{\tau}[(p_0+i\sigma)^2-1]\times\left[1-\frac{x}{ct(p_0+i\sigma)}\right]\right\}\times(p_0+i\sigma)i\,d\sigma.\tag{38}$$

We can then expand and approximate the exponential to obtain

$$u(x,t)\approx u_0\frac{p_0}{\pi\tau}\int_{C'}\exp\left[f(p_0)-\frac{t}{\tau}\sigma^2\left(1+\frac{\alpha}{p_0^3}\right)\right]d\sigma,\tag{39}$$

where $\alpha = x/(ct)$. Thus,

$$u(x,t) \approx u_0 \frac{p_0 e^{f(p_0)}}{\pi\tau} \int_{-\infty}^{+\infty} e^{-A\sigma^2} d\sigma, \tag{40}$$

where

$$f(p_0) = \frac{t}{\tau}(p_0^2 - 1)\left(1 - \frac{\alpha}{p_0}\right) \tag{41}$$

and p_0 is the root of $f'(p) = 0$ between $p = 1$ and $p = \alpha$. Since the integral in Eq. (40) has the known value $\sqrt{\pi/A}$, we obtain as the final result

$$u(x,t) \approx u_0 \frac{p_0 e^{f(p_0)}}{\sqrt{\pi\tau t}} \sqrt{\frac{p_0^3}{\alpha + p_0^3}} \quad (\alpha > 1). \tag{42}$$

The dependence on x appears only through α, which is required to find p_0. This solution is singular at $t = 0$, which is the time at which the impulse $\delta(t)$ acts.

Equation (42) is also valid for $\alpha < 1$. In that case, C' does not cross the essential singularity at $p = 0$. However, Eq. (42) is not valid for $\alpha = 1$. For this case, Eq. (34) implies

$$2p^3 - p^2 - 1 = 0, \tag{43}$$

which has one real root $p_0 = 1$ and two complex roots $(-1 \pm i\sqrt{3})/4$. It is convenient for this case to use the changes of variable

$$p = 1 + v, \tag{44}$$

which will have the effect of introducing a shift, and

$$t = t' + \frac{x}{c}. \tag{45}$$

With these changes of variables, Eq. (30) becomes

$$u(x,t) = u_0 \frac{1}{\pi i \tau} \int_L \exp\left[\frac{1}{\tau}(v^2 + 2v) \times \left(t' + \frac{x}{c} - \frac{x}{c(1+v)}\right)\right] \times (1+v)\, dv. \tag{46}$$

For $v \ll 1$, this integral simplifies to

$$u(x,t) \approx u_0 \frac{1}{2\pi i \tau} \int_{-i\infty}^{+i\infty} \exp\left[\frac{2v}{\tau}\left(t' + \frac{2xv}{c}\right)\right] dv. \tag{47}$$

With the additional change of variable

$$v = \frac{\sqrt{c\tau p}}{2}, \tag{48}$$

Equation (47) becomes

$$u(x,t) \approx u_0 \frac{1}{2\pi i} \cdot \frac{\sqrt{c\tau}}{2\tau} \int_{-i\infty}^{+i\infty} \exp\left(xp + t' \sqrt{\frac{cp}{\tau}}\right) \frac{dp}{\sqrt{p}}, \tag{49}$$

which is in the form of an inverse Laplace transform in x [15,13]. Thus,

$$u(x,t) \approx u_0 \frac{1}{2} \sqrt{\frac{c}{\tau}} \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{p}} e^{-k\sqrt{p}}\right\}, \tag{50}$$

where

$$k = -t' \sqrt{\frac{c}{\tau}}. \tag{51}$$

Since this transform is tabulated [12], we obtain

$$u(x,t) \approx u_0 \sqrt{\frac{c}{4\pi\tau x}} \exp\left(-\frac{ct'^2}{4\tau x}\right). \tag{52}$$

However, since $\alpha = 1$ (and $x = ct$), it therefore follows that

$$u(x,t) \approx u_0 \frac{e^{-t/\tau}}{\sqrt{4\pi\tau t}} \exp\left(-\frac{x^2}{4c^2\tau t}\right), \tag{53}$$

where we have included the $e^{-t/\tau}$ factoring resulting from the "shift." This result coincides with the first term of Eq. (26) with $x = ct$ in the amplitude factor. Thus, the first two terms of the earlier solution are the strongest contributors to the inversion integral [[16], Chap. 4].

Dispersion Relation

If we seek a solution of Eq. (1a) in the form

$$u(x,t) = e^{i(kx - \omega t)}, \tag{54}$$

where $k = \omega/c$, we obtain the quadratic

$$\omega^2 + (i\nu k^2)\omega - k^2 c^2 = 0, \tag{55}$$

whose solutions are

$$\omega_{1,2} = \pm k\sqrt{c^2 - (\nu k/2)^2} - i\nu k^2/2. \tag{56}$$

If $c > \nu k/2$ (or $\omega\tau < 2$), the solutions are complex with a negative imaginary part, as illustrated in Fig. 2. In this case, there is wave propagation through the medium with an attenuation factor $e^{-\nu k^2 t/2}$, which reduces the wave amplitude with time. The second possibility, $c < \nu k/2$ (or $\omega\tau > 2$), is of no interest, since the ω solutions are purely imaginary, and there is no wave propagation. This situation then resembles ordinary diffusion.

Equation (55) can also be written in terms of $\tau = \nu/c^2$, in which case

$$\omega^2 = (kc)^2(1 - i\omega\tau), \tag{57}$$

which can be solved for k:

$$k = \frac{\omega}{c\sqrt{1 + (\omega\tau)^2}} \sqrt{1 + i\omega\tau} \tag{58}$$

$$= \frac{\omega}{c\sqrt{2}\sqrt{1 + (\omega\tau)^2}} \left[\sqrt{\sqrt{1 + (\omega\tau)^2} + 1} + i\sqrt{\sqrt{1 + (\omega\tau)^2} - 1}\right]. \tag{59}$$

If we now express this complex propagation constant in the usual form

$$k = \frac{\omega}{c_p} + i\alpha, \tag{60}$$

where c_p is the phase velocity, and α is the attenuation, we obtain

$$c_p = \frac{c\sqrt{2}\sqrt{1 + (\omega\tau)^2}}{\sqrt{\sqrt{1 + (\omega\tau)^2} + 1}}, \tag{61}$$

$$\alpha = \frac{\omega}{c\sqrt{2}} \cdot \frac{\sqrt{\sqrt{1 + (\omega\tau)^2} - 1}}{\sqrt{1 + (\omega\tau)^2}}. \tag{62}$$

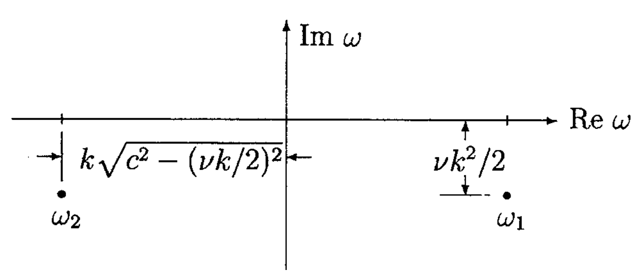

Fig. 2 The ω solutions of the dispersion relation

We can also write the attenuation factor α in terms of the phase velocity:

$$\alpha=\frac{\sqrt{1+(\omega\tau)^2}-1}{c_p\tau}. \tag{63}$$

We state for completeness that, for harmonic processes, the solution of Eq. (1) can be written as

$$u(x,t)=e^{-\alpha x}e^{-i\omega(t-x/c_p)}, \tag{64}$$

where α and c_p are given in Eqs. (63) and (62), respectively. In the common case of weak absorption, the pertinent region of validity reduces from $\omega\tau<2$ to $\omega\tau\ll 1$, and then $k\rightarrow\omega/c$, and $\alpha\rightarrow\omega^2\tau/(2c)$, which are in agreement with Malecki's results [3] expressed in a different form. One could also see how the solution in Eq. (64) could be used to recover the impulsive excitation at the origin in the transient case discussed initially.

Numerical Results and Discussion

We have obtained both exact and approximate solutions for the propagation behavior of a delta function pulse that travels in a viscous and dispersive medium obeying a general boundary and initial value problem governed by a third-order partial differential equation. The exact solution was obtained by operational methods, where the resulting infinite series was expressed in terms of repeated integrals of the complementary error function. It was also shown how to transform this series into a rapidly convergent power series.

The approximate solution was found independently by the method of stationary phase. It agrees exactly with the first term of the series solution found before.

We return to Eq. (52), which was already seen to agree with the first term of the exact series solution. Using the shift in Eq. (45) and changing our notation for τ to that of Lagerstrom [9] yields

$$u(x,t)\approx\frac{u_0}{\sqrt{2\pi t(4\nu/3c^2)}}\exp\left[-\frac{(x-ct)^2}{8\nu t/3}\right]. \tag{65}$$

We compare this term to the asymptotic expansion for the inversion integral, Eq. (6), given by Lagerstrom [9] [i.e., Eq. (2.212*a*), p. 50] for large values of *t*:

$$u(x,t)\underset{t\rightarrow\infty}{\rightarrow}u_0\sqrt{\frac{4\nu/3c^2}{2\pi t}}\exp\left[-\frac{(x-ct)^2}{8\nu t/3}\right]+\ldots. \tag{66}$$

We note that, in [9], the factor $4\nu/3c^2$ has been erroneously placed in the numerator of the radical. It follows that the asymptotic expansion for the inversion integral in [9] can also be obtained from the first term of our complete analytic solution after the correction of the minor algebraic error and deleting the $e^{-t/\tau}$ factor. It is also well-known that a function of the form

$$\frac{u_0/c}{\sqrt{4\pi\tau t}}e^{-(x-x')^2/(4c^2\tau t)}$$

satisfies the heat equation

$$u_t=c^2\tau u_{xx} \tag{67}$$

with $x'=ct$. It is then clear why a solution of the heat equation is a good approximation to the solution of Eq. (1) for large *t* [6].

An asymptotic approximation of the inversion integral for *small t* can also be extracted from the second term of our series solution. We rewrite it, in the notation of [9], as

$$u(x,t)\underset{t\rightarrow 0}{\rightarrow}\frac{u_0x}{\sqrt{2\pi(4\nu/3)}t^{3/2}}\exp\left[-\frac{x^2}{2(4\nu/3)t}\right]+\ldots, \tag{68}$$

a result which does seem to be available elsewhere. The two asymptotic expansions in Eqs. (65) and (68) have the same structure and have amplitudes proportional to $t^{-1/2}$ and its derivative $t^{-3/2}$ for large and small *t*, respectively. The double series in the solution, Eq. (26), seems to affect the intermediate region away from $t=0$ or $t=\infty$ only slightly.

We display these results in several plots involving the nondimensional solution presented in Eq. (25). In Fig. 3 is shown a contour plot of the first two terms (omitting the double sum) of the expression for the nondimensional fluid particle velocity $\bar{u}$ (denoted *ubar* in the figure) vs. nondimensional distance *xbar* from the origin and nondimensional time *tbar*. In Figs. 4 and 5, we show various cuts taken from Fig. 3. In Fig. 4, nondimensional velocity is plotted vs. nondimensional time for several values of leading nondimensional distance. In Fig. 5, nondimensional velocity is plotted vs. nondimensional distance for several values of

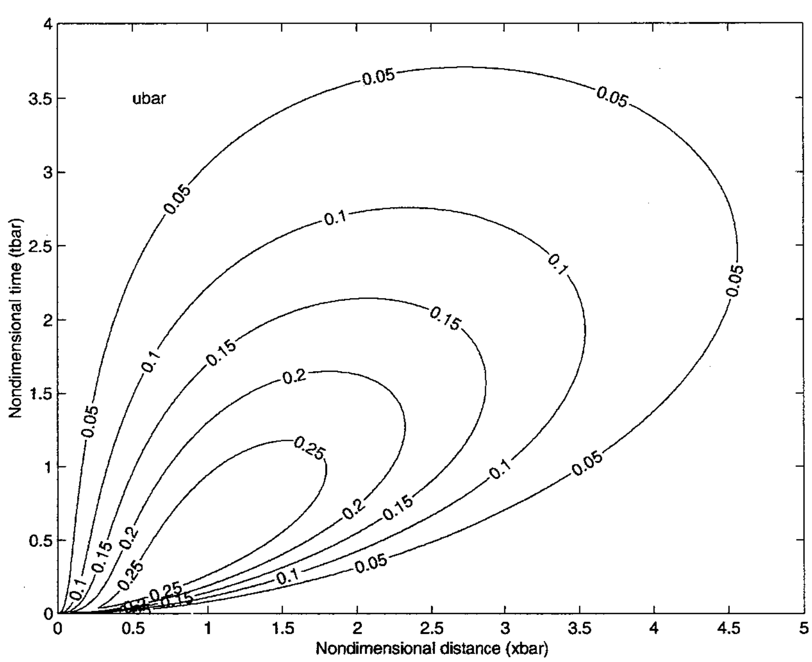

Fig. 3 Contour plot of nondimensional velocity vs. nondimensional distance and time. The numbers along the curves are the constant values of "ubar" along each curve.

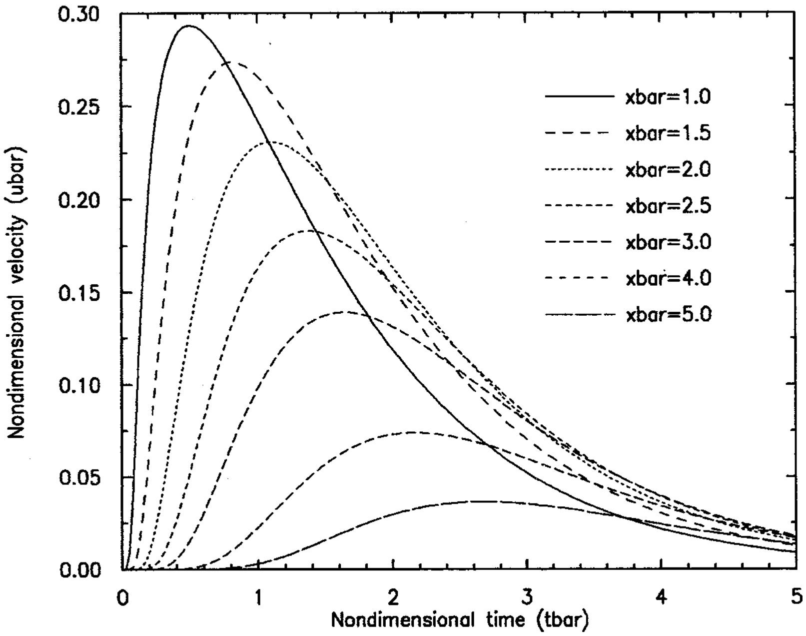

Fig. 4 Plot of the nondimensional velocity (ubar) vs. nondimensional time (tbar) for several values of the nondimensional distance (xbar).

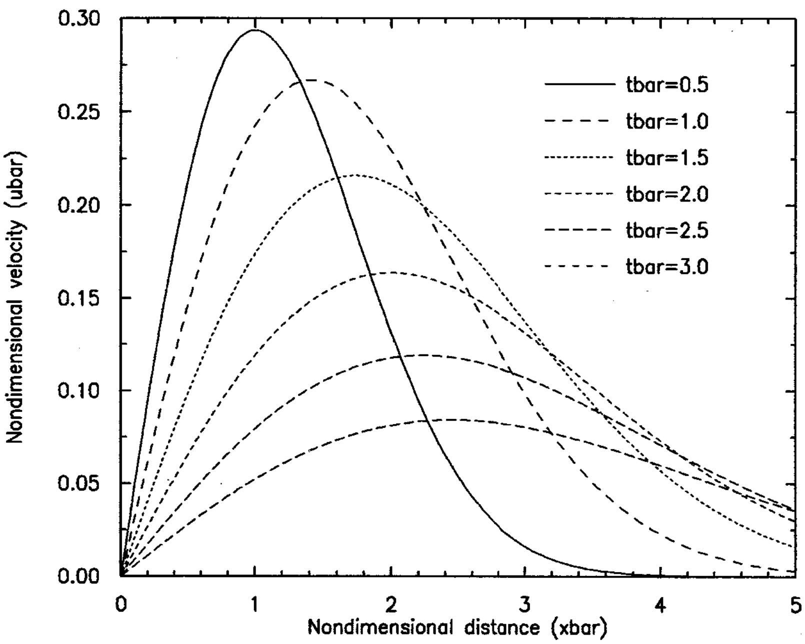

Fig. 5 Plot of nondimensional velocity vs. nondimensional distance for several values of nondimensional time

nondimensional time. All three plots show that, as the initial impulse advances either in time or space, its peak amplitude decays, and its width broadens, as would be expected qualitatively. However, the formulas derived (along with their corresponding plots) represent a *quantitative* evaluation of the anticipated effects of viscosity and dispersion.

Thus, we have analytically and quantitatively described the decay and broadening of impulses propagating in viscous media as modeled by Stokes' classical boundary-initial value problem. In the process, we have obtained asymptotic expansions of the solution valid for both large and small time.

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