Energy radiated by a point acoustic dipole that reverses its uniform velocity along its rectilinear path

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This work extends a mathematical approach developed recently for monopoles to describe the sound energy radiated by a rectilinearly moving dipole that changes direction along its trajectory. Although the dipole travels with constant speed, it undergoes acceleration by reversing its direction during a finite time interval along its path. This work determines the joint angular and frequency distribution of the radiated energy, its angular distribution, and the total radiated energy output. Results for the radiated energy are systematized by expressing the radiation integrals in terms of hypergeometric functions. This procedure simplifies the evaluations, particularly at low Mach numbers, and permits the comparison of results to the earlier monopole case. [DOI: 10.1121/1.1532031]

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I. INTRODUCTION

There are various situations involving the rectilinear motion of multipole sources that generate still unexamined radiation fields. Those situations in which the multipole undergoes acceleration only through a finite portion of its straight path are of particular interest. Some of these situations were considered in a recent paper[1] for monopole sources. This paper extends the mathematical analysis of a point acoustic dipole that reverses direction along its rectilinear path.

A traditional model for the sound field radiated by a rigid body moving slowly and uniformly through a fluid is that of a moving point dipole with the dipole axis in the direction of motion.[2,3] Today it is known that such a model is too simple[4] and requires correction. The effect of motion is far more complicated, and that effect does not involve just Doppler factors.[5–7] It has been shown that there is amplification in a direction normal to the source motion and that there is an additional omnidirectional term in the sound field. However, analyses that follow the traditional model are still very useful and form a good basis for the investigation of more complicated situations. Such situations can be decomposed into simpler cases such as the point-dipole analysis presented here. There are moving bodies that can reverse their rectilinear direction of motion in very small distances.[1] The next section attempts to provide an analytical prediction for the sound energy radiated during one such turn-around maneuver without dealing with the corrections introduced by the finite size of these bodies.

II. ENERGY RADIATION FROM A POINT DIPOLE WITH A FINITE PERIOD OF UNSTEADY RECTILINEAR MOTION

Consider a dipole source of strength α moving in a rectilinear path, as shown in Fig. 1. The motion starts at $x=-\infty$ when $t=-\infty$. It then moves forward along the x-direction with constant speed v_0. It subsequently slows down to zero speed when $t=0$ at $x=0$, and finally, it turns around and reverses its earlier motion and continues to $x=-\infty$ for $t=\infty$ with the same constant speed v_0. Since the rectilinear motion is in one dimension, this figure shows two (upper and lower) paths of motion for clarity only.

The position $x_s(t)$, speed $v_s(t)$, and acceleration $\dot{v}_s(t)$ of a point dipole which moves along the half-line $-\infty<x<0$ are given, respectively, by

$$x_s(t)=\begin{cases} v_0(t+\tau-2\tau/\pi), & -\infty<t<-\tau, \\ 2v_0(\tau/\pi)[\cos(\pi t/2\tau)-1], & -\tau<t<\tau, \\ v_0(-t+\tau-2\tau/\pi), & \tau<t<\infty, \end{cases} \tag{1}$$

$$v_s(t)=\begin{cases} v_0, & -\infty<t<-\tau, \\ -v_0\sin(\pi t/2\tau), & -\tau<t<\tau, \\ -v_0, & \tau<t<\infty, \end{cases} \tag{2}$$

$$\dot{v}_s(t)=\begin{cases} 0, & -\infty<t<-\tau, \\ -[v_0\pi/(2\tau)]\cos(\pi t/2\tau), & -\tau<t<\tau, \\ 0, & \tau<t<\infty. \end{cases} \tag{3}$$

Sound radiation will occur for $|t|<\tau$, which is the time interval in which the dipole changes direction, and there is a nonvanishing acceleration.

The governing inhomogeneous wave equation in this case is

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{r},t)=-\frac{1}{\rho_0}Q(\mathbf{r},t)$$
$$=-\frac{\alpha}{\rho_0}\dot{x}_s(t)\frac{\partial}{\partial x}\delta[x-x_s(t)], \tag{4}$$

where c is the sound speed, ρ_0 is the equilibrium density of the medium, and δ is the Dirac delta function (see Appendix

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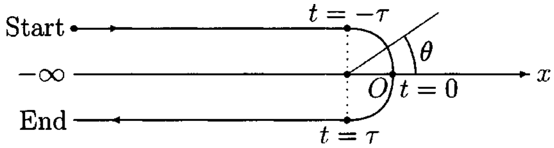

FIG. 1. An acoustic point dipole in rectilinear motion exhibiting a change of direction around $t=0$.

A for more details). The directional distribution of the radiated energy $\tilde{E}(\mathbf{n})$ is known[1] to be

$$\tilde{E}(\mathbf{n})=\int_{-\infty}^{\infty} E(\mathbf{n},\omega)\,d\omega, \tag{5}$$

where ω is the circular frequency of the dipole, and $E(\mathbf{n},\omega)$ is the directional distribution of the radiated energy spectrum given by[1,8–10]

$$E(\mathbf{n},\omega)=\frac{\omega^2}{2\pi\rho_0 c(4\pi)^2}\left|\int_{-\infty}^{\infty} Q(\mathbf{r},t)e^{i\omega(t-\mathbf{n}\cdot\mathbf{r}/c)}\,d\mathbf{r}\,dt\right|^2. \tag{6}$$

Here $\mathbf{n}$ is an arbitrary unit vector in some direction.

The corresponding radiation field integral in the above equation, henceforth denoted J_R, is obtained by substituting $Q(\mathbf{r},t)$ from Eq. (4):

$$J_R=\int_{-\infty}^{\infty} Q(\mathbf{r},t)e^{i\omega(t-\mathbf{n}\cdot\mathbf{r}/c)}\,d\mathbf{r}\,dt \tag{7}$$

$$=\alpha\int_{-\infty}^{\infty}\dot{x}_s(t)e^{i\omega t}\,dt\int_{-\infty}^{\infty}\frac{\partial}{\partial x}[\delta(x-x_s(t))]e^{-i\omega\mathbf{n}\cdot\mathbf{r}/c}\,d\mathbf{r} \tag{8}$$

$$=i\alpha\frac{\omega}{c}\cos\theta\int_{-\infty}^{\infty}\dot{x}_s(t)e^{i\omega[t-(\cos\theta x_s(t)/c)]}\,dt. \tag{9}$$

This integration can be split into three regimes similar to those in a recent paper,[1] viz.,

$$J_R=v_0\int_{-\infty}^{-\tau}e^{i\omega[t-M\cos\theta(t+\tau-2\tau/\pi)]}\,dt \tag{10}$$

$$-v_0\int_{-\tau}^{\tau}\sin(\pi t/2\tau)e^{i\omega[t-(2M\tau/\pi)\cos\theta(\cos(\pi t/2\tau)-1)]}\,dt \tag{11}$$

$$-v_0\int_{\tau}^{\infty}e^{i\omega[t-M\cos\theta(-t+\tau-2\tau/\pi)]}\,dt, \tag{12}$$

where the Mach number is $M=v_0/c$. The above integrals lead to expressions containing delta functions, which can then be discarded by noticing that $\omega\delta(\omega)=0$. Then, after integrating by parts, we obtain

$$J_R=-\frac{\pi v_0}{2\omega^2\tau}e^{2i(\omega\tau/\pi)M\cos\theta}\times\int_{-\pi/2}^{\pi/2}e^{(2i\omega\tau/\pi)(\sigma-M\cos\theta\cos\sigma)}\times\frac{\sin\sigma+M\cos\theta(1+2\cos^2\sigma)}{(1+M\cos\theta\sin\sigma)^4}\,d\sigma, \tag{13}$$

where the normalized time $\sigma=\pi t/(2\tau)$ has been introduced.

TABLE I. Cases plotted in the figures.

Location	Time t	Right side of Eq. (16)	σ	Figure
Before change in direction	-2τ	$9M^4\cos^4\theta$	$-\pi$	2
At start of change	$-\tau$	$M^2\cos^2\theta/(1-M\cos\theta)^7$	$-\pi/2$	3
Middle of change	0	$9M^4\cos^4\theta$	0	2
End of change	τ	$M^2\cos^2\theta/(1+M\cos\theta)^7$	$\pi/2$	4
After change in direction	2τ	$9M^4\cos^4\theta$	π	2

Substitution of this result into Eq. (6) then yields $E(\mathbf{n},\omega)$ in the form

$$E(\mathbf{n},\omega)=\frac{\alpha^2\cos^2\theta}{2\pi\rho_0c^3(4\pi)^2}\frac{\pi^2v_0^2}{4\tau^2}\times\left|\int_{-\pi/2}^{\pi/2}e^{(2i\omega\tau/\pi)(\sigma-M\cos\theta\cos\sigma)}\times\frac{\sin\sigma+M\cos\theta(1+2\cos^2\sigma)}{(1+M\cos\theta\sin\sigma)^4}\,d\sigma\right|^2. \tag{14}$$

Integration over all frequencies eventually yields the angular distribution of the radiated energy from the moving dipole, viz.,

$$\check{E}(\theta)=\frac{\pi\alpha^2M^2\cos^2\theta}{128\rho_0c\tau^3}\times\int_{-\pi/2}^{\pi/2}\frac{[\sin\sigma+M\cos\theta(1+2\cos^2\sigma)]^2}{(1+M\cos\theta\sin\sigma)^9}\,d\sigma. \tag{15}$$

The joint angular and temporal distribution of the radiated energy, which could also be called the instantaneous angular distribution of the radiated energy, is denoted $\hat{E}(\theta,\sigma)$ and is given in nondimensional form by

$$\frac{128\rho_0c\tau^3}{\pi\alpha^2}\hat{E}(\theta,\sigma)=M^2\cos^2\theta\times\frac{[\sin\sigma+M\cos\theta(1+2\cos^2\sigma)]^2}{(1+M\cos\theta\sin\sigma)^9}. \tag{16}$$

Since there is a nonvanishing acceleration only within the interval $-\tau<t<\tau$, integration of Eq. (16) over σ from $-\pi/2$ to $\pi/2$ recovers the result indicated in Eq. (15) for the angular distribution $\check{E}(\theta)$. Since the evaluation of this general result, as seen below, is quite cumbersome, Table I illustrates some particular relevant cases (also see Fig. 1).

These angular patterns are displayed in Figs. 2, 3, and 4 for Mach numbers from $M=0.10$ to $M=0.25$. Note the different scales used in the figures. Also note that Figs. 2, 3, and 4 are mirror reflections of each other about the plane $\theta=\pi$, which is an indication that the forward (or backward) radiation lobes reverse positions at the start and end of the acceleration-change interval. At their peaks, the lobes in Figs. 3 and 4 have amplitudes about an order of magnitude larger than either before or after the acceleration underwent changes (Fig. 2), in agreement with expectations.

An integration of Eq. (16) over θ yields $\bar{E}(\sigma)$, which has three terms:

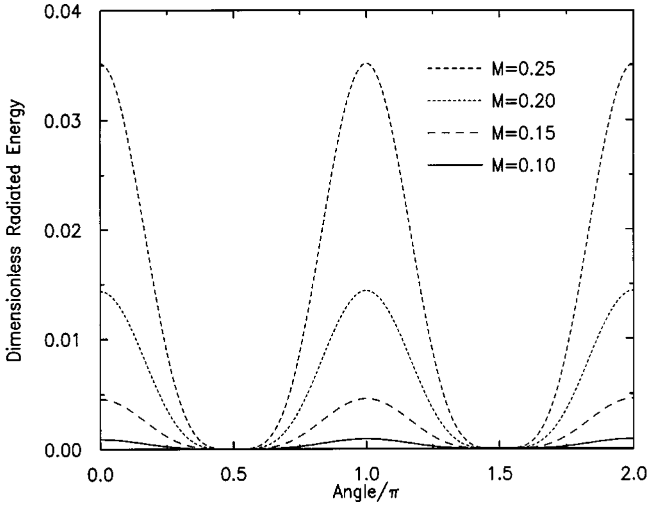

FIG. 2. Angular distribution of the instantaneously radiated energy from the moving dipole at time $t=-2\tau$, 0, and 2τ for Mach numbers from $M=0.10$ to $M=0.25$.

$$
\begin{aligned}
\frac{128\rho_0 c\tau^3}{\pi\alpha^2}\overline{E}(\sigma) &= 2\pi\int_0^{\pi}\frac{128\rho_0 c\tau^3}{\pi\alpha^2}\hat{E}(\theta,\sigma)\sin\theta\, d\theta \\
&= 2\pi M^2\sin^2\sigma\int_0^{\pi}\frac{\cos^2\theta\sin\theta\, d\theta}{(1+M\cos\theta\sin\sigma)^9} \\
&\quad + 4\pi M^3\sin\sigma(3-2\sin^2\sigma) \\
&\quad \times\int_0^{\pi}\frac{\cos^3\theta\sin\theta\, d\theta}{(1+M\cos\theta\sin\sigma)^9} \\
&\quad + 2\pi M^4(3-2\sin^2\sigma)^2 \\
&\quad \times\int_0^{\pi}\frac{\cos^4\theta\sin\theta\, d\theta}{(1+M\cos\theta\sin\sigma)^9}. \qquad (17)
\end{aligned}
$$

The transformation $x=\cos\theta$ and the relabeling $N=-M\sin\sigma$ recasts Eq. (17) in the form

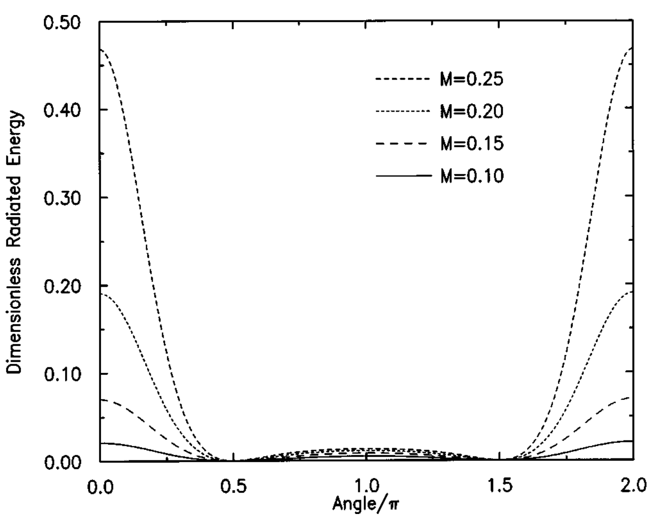

FIG. 3. Angular distribution of the instantaneously radiated energy from the moving dipole at time $t=-\tau$, when the dipole begins its turn-around maneuver, for Mach numbers from $M=0.10$ to $M=0.25$.

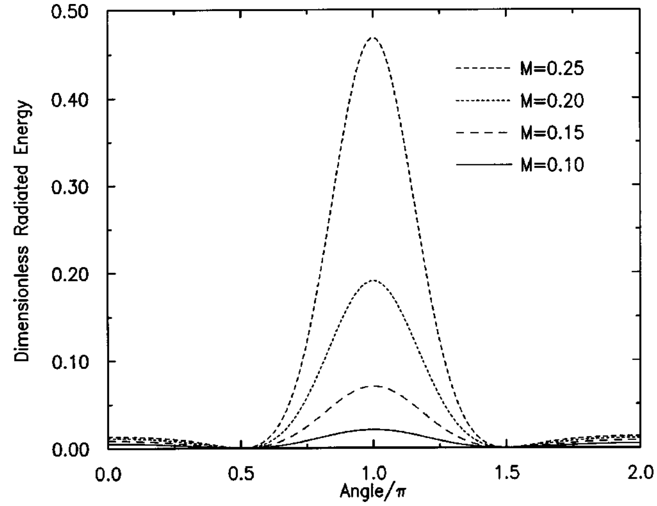

FIG. 4. Angular distribution of the instantaneously radiated energy from the moving dipole at time $t=\tau$, when the dipole ends its turn-around maneuver, for Mach numbers from $M=0.10$ to $M=0.25$.

$$
\begin{aligned}
\frac{128\rho_0 c\tau^3}{\pi\alpha^2}\overline{E}(\sigma) &= 2\pi M^2\sin^2\sigma\int_{-1}^{1}\frac{x^2\, dx}{(1-Nx)^9} \\
&\quad + 4\pi M^3\sin\sigma(3-2\sin^2\sigma) \\
&\quad \times\int_{-1}^{1}\frac{x^3\, dx}{(1-Nx)^9} + 2\pi M^4(3-2\sin^2\sigma)^2 \\
&\quad \times\int_{-1}^{1}\frac{x^4\, dx}{(1-Nx)^9}. \qquad (18)
\end{aligned}
$$

These three integrals can be evaluated exactly, but such evaluation leads to quite cumbersome expressions. For example, it has been shown[11] that, for any polynomial $P(x)$ of degree less than n,

$$
\begin{aligned}
\int_{-1}^{1}\frac{P(x)dx}{(1-Nx)^n} &= \sum_{k=0}^{n-2}A_k\left[\frac{1}{(1-N)^{n-k-1}}-\frac{1}{(1+N)^{n-k-1}}\right] \\
&\quad + A_{n-1}\ln\left(\frac{1-N}{1+N}\right), \qquad (19)
\end{aligned}
$$

where

$$
A_k=\frac{(-1)^{k+1}}{N^{k+1}k!}\left[\frac{d^kP(x)}{dx^k}\right]_{x=1/N}, \quad k=0,1,2,\ldots,n-1. \qquad (20)
$$

The application of this result to the last of the integrals in Eq. (18) yields the expression

$$
\begin{aligned}
\int_{-1}^{1}\frac{x^4\, dx}{(1-Nx)^9} &= \frac{1}{N^5}\left\{\frac{1}{8}\left[\frac{1}{(1-N)^8}-\frac{1}{(1+N)^8}\right]\right. \\
&\quad -\frac{4}{7}\left[\frac{1}{(1-N)^7}-\frac{1}{(1+N)^7}\right]+\left[\frac{1}{(1-N)^6}\right. \\
&\quad \left. -\frac{1}{(1+N)^6}\right]-\frac{4}{5}\left[\frac{1}{(1-N)^5}-\frac{1}{(1+N)^5}\right] \\
&\quad \left. +\frac{1}{4}\left[\frac{1}{(1-N)^4}-\frac{1}{(1+N)^4}\right]\right\}. \qquad (21)
\end{aligned}
$$

Similar results could be obtained for the other two integrals. Since these exact expressions are not easily amenable to expansion in powers of N, we take instead a different approach.

Since $N=-M\sin\sigma$, it is clear that $\overline{E}(\sigma)$, in general, is a very complicated function of σ. However, these integrals can also be evaluated in terms of a particular form of the Gauss hypergeometric function

$$\int_0^1 \frac{x^{s-1}\,dx}{(1+\beta x)^n}=\frac{1}{s}F(n,s;1+s;-\beta), \tag{22}$$

where $F(a,b;c;x)$ is the Gauss hypergeometric function.[12] The last of the integrals in Eq. (18) is then

$$\begin{aligned}\int_{-1}^1 \frac{x^4\,dx}{(1-Nx)^9}&=\int_0^1 \frac{x^4\,dx}{(1+Nx)^9}+\int_0^1 \frac{x^4\,dx}{(1-Nx)^9}\\&=\tfrac{1}{5}[F(9,5;6;-N)+F(9,5;6;N)]\\&\approx\tfrac{2}{5}(1+\tfrac{225}{7}N^2+\cdots).\end{aligned} \tag{23}$$

Similarly, the other two integrals reduce to

$$\int_{-1}^1 \frac{x^3\,dx}{(1-Nx)^9}\approx\frac{1}{2}(1+30N^2+\cdots), \tag{24}$$

$$\int_{-1}^1 \frac{x^2\,dx}{(1-Nx)^9}\approx\frac{2}{3}(1+27N^2+\cdots), \tag{25}$$

where the hypergeometric functions have been expanded in terms of the standard hypergeometric series.[12,13]

It then follows from Eq. (18) that

$$\begin{aligned}\frac{128\rho_0 c\tau^3}{\pi\alpha^2}\overline{E}(\sigma)=&\frac{4\pi}{3}M^2\sin^2\sigma(1+27M^2\sin^2\sigma+\cdots)\\&+2\pi M^3\sin\sigma(3-2\sin^2\sigma)\\&\times(1+30M^2\sin^2\sigma+\cdots)+\frac{4\pi}{5}M^4\\&\times(3-2\sin^2\sigma)^2(1+\tfrac{225}{7}M^2\sin^2\sigma+\cdots)\\&+\cdots.\end{aligned} \tag{26}$$

Since $\sigma=\pi t/(2\tau)$, Eq. (26) can also be viewed as describing the complicated temporal distribution or instantaneous energy radiated at time t by the dipole. To verify this interpretation, we note that, for $t=\tau$, for example, one has $\sigma=\pi/2$, in which case Eq. (26) reduces (up to order M^2) to

$$\frac{128\rho_0 c\tau^3}{\pi\alpha^2}\overline{E}\left(\frac{\pi}{2}\right)\approx\frac{4\pi}{3}M^2. \tag{27}$$

The angular distribution entry in Table I for $t=\tau$ is

$$\frac{128\rho_0 c\tau^3}{\pi\alpha^2}E(t=\tau)=\frac{M^2\cos^2\theta}{(1+M\cos\theta)^7}. \tag{28}$$

If we then integrate over all θ and retain only terms of $O(M^2)$, we find that

$$\begin{aligned}\frac{128\rho_0 c\tau^3}{\pi\alpha^2}E(t=\tau)&=2\pi M^2\int_0^\pi \frac{\cos^2\theta\sin\theta\,d\theta}{(1+M\cos\theta)^7}\\&\approx 4\pi M^2/3,\end{aligned} \tag{29}$$

which agrees with Eq. (27). For slow motions (i.e., $M^2\ll 1$), Eq. (26) reduces to its first term, in which case, for any t,

$$\overline{E}\left(\frac{\pi t}{2\tau}\right)=\frac{\pi^2\alpha^2M^2}{96\rho_0 c\tau^3}\sin^2\left(\frac{\pi t}{2\tau}\right). \tag{30}$$

Returning to the direct evaluation of Eq. (15), the evaluation of the total energy output from the motion is the most important result. Its explicit evaluation is possible based on the double integral

$$I=\int_0^\pi \sin\theta\cos^2\theta\,d\theta\int_{-\pi/2}^{\pi/2}\frac{\sin^2\sigma+2M\cos\theta\sin\sigma(3-2\sin^2\sigma)+M^2\cos^2\theta(3-2\sin^2\sigma)^2}{(1+M\cos\theta\sin\sigma)^9}\,d\sigma, \tag{31}$$

which can be split into the three integrals

$$I=I_1+I_2+I_3, \tag{32}$$

where

$$I_1=\int_0^\pi \sin\theta\,d\theta\int_{-\pi/2}^{\pi/2}\frac{\cos^2\theta\sin^2\sigma}{(1+M\cos\theta\sin\sigma)^9}\,d\sigma, \tag{33}$$

$$\begin{aligned}I_2/(2M)=&\int_0^\pi \sin\theta\cos^2\theta\,d\theta\int_{-\pi/2}^{\pi/2}(3-2\sin^2\sigma)\\&\times\frac{\cos\theta\sin\sigma}{(1+M\cos\theta\sin\sigma)^9}\,d\sigma,\end{aligned} \tag{34}$$

$$I_3/M^2=\int_0^\pi \sin\theta\cos^4\theta\,d\theta\int_{-\pi/2}^{\pi/2}\frac{(3-2\sin^2\sigma)^2}{(1+M\cos\theta\sin\sigma)^9}\,d\sigma. \tag{35}$$

The last three integrals (I_1, I_2, I_3) can be reduced to the sum of simpler integrals by successive integrations by parts, the details of which are given in Appendix B. The result of this simplification is

$$I = I_1 + I_2 + I_3 = \frac{2\pi}{3}\left(1 + \frac{621}{40}M^2 + \cdots\right), \tag{36}$$

and the total energy radiated by the dipole is

$$\begin{aligned} E_{\text{dip}} &= 2\pi \int_0^{\pi} \check{E}(\theta)\sin\theta\, d\theta \\ &= \frac{\pi^2 M^2 \alpha^2}{128\rho_0 c \tau^3} I \\ &= \frac{\pi^3 M^2 \alpha^2}{192\rho_0 c \tau^3}\left(1 + \frac{621}{40}M^2 + \cdots\right), \quad M \ll 1. \end{aligned} \tag{37}$$

It would be possible to obtain more terms by keeping them in the expression for the hypergeometric series. For small Mach numbers, the above result suffices. For a very slow motion limit (i.e., $M \to 0$), the above result can be compared to that of a monopole source of strength q undergoing the same motion.[1] This result was

$$E_{\text{mon}} \to \frac{q^2 M^2 \pi}{48\rho_0 c \tau}. \tag{38}$$

Therefore, the ratio of the radiated energies in these two cases is

$$\lim_{M\to 0} \frac{E_{\text{dip}}}{E_{\text{mon}}} = \left(\frac{\pi\alpha}{2q\tau}\right)^2. \tag{39}$$

Thus, the ratio of radiated energies is directly proportional to the square of the ratio of their strengths and inversely proportional to the square of the finite time-span during which both multipoles are changing direction.

Additional comparisons of methods used here are possible now. For a slow motion (i.e., $M \ll 1$), the right-hand side of Eq. (26) reduces to its first term. Integration over σ then yields

$$\frac{128\rho_0 c \tau^3}{\pi\alpha^2} E_{\text{dip}} \underset{M\ll 1}{\to} \frac{4\pi}{3} M^2 \int_{-\pi/2}^{\pi/2} \sin^2\sigma\, d\sigma = 2\pi^2 M^2/3 \tag{40}$$

or

$$E_{\text{dip}} \to \frac{\pi^3 M^2 \alpha^2}{192\rho_0 c \tau^3}, \tag{41}$$

which agrees with the approach that led to Eq. (37) for small M. It was found earlier that, for $t = 0$, Eq. (16) produced the angular pattern

$$\frac{128\rho_0 c \tau^3}{\pi\alpha^2} \hat{E}(\theta, \sigma = 0) = 9M^4 \cos^2\theta, \tag{42}$$

and when this expression is integrated over all θ, the resulting energy is

$$E_{\text{dip}} = 2\pi \int_0^{\pi} 9M^4 \cos^4\theta \sin\theta\, d\theta = 36\pi M^4/5. \tag{43}$$

In this same case, the method that led to Eq. (26) yields

$$E_{\text{dip}} \underset{\sigma\to 0}{\to} 2\pi\left(\tfrac{2}{5}\right) M^4 \cdot 9 = 36\pi M^4/5, \tag{44}$$

which is also in agreement with the previous method above.

III. CONCLUDING REMARKS

The angular distribution of the energy spectrum, the angular distribution of the radiated acoustic energy, and the total radiated energy output of the dipole undergoing the rectilinear motion described in Eq. (1) have been obtained. The evaluation of the integrals is performed in terms of hypergeometric functions. The results are approximated for low Mach number values. A comparison with an analogous result for a monopole undergoing the same ‘‘partially accelerated’’ motion is obtained. Some angular distribution patterns of the radiated energy are plotted for some particular cases.

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APPENDIX A

If $\mathbf{n}$ denotes an arbitrarily oriented unit vector and $d\Omega_n$ the element of solid angle about $\mathbf{n}$, then the relations[1,8–11]

$$P(t) = \int Q(\mathbf{r},t)\frac{\partial\phi}{\partial t} d\mathbf{r} = \int P(\mathbf{n},t)\, d\Omega_n \tag{A1}$$

and

$$\begin{aligned} P(\mathbf{n},t) = &-\frac{1}{16\rho_0 c \pi^2}\int_{-\infty}^{\infty} Q(\mathbf{r},t) Q(\mathbf{r}',t') \\ &\times \delta''\left(t' - t + \frac{\mathbf{n}}{c}\cdot(\mathbf{r}-\mathbf{r}')\right) d\mathbf{r}\, d\mathbf{r}'\, dt' \end{aligned} \tag{A2}$$

specify the overall instantaneous power $P(t)$ radiated from the source into its surroundings and the amount $P(\mathbf{n},t)$ radiated along the given $\mathbf{n}$ direction. It then follows that the expressions

$$\begin{aligned} E(\mathbf{n}) &= \int_{-\infty}^{\infty} P(\mathbf{n},t)\, dt \\ &= -\frac{1}{16\rho_0 c \pi^2}\int_{-\infty}^{\infty} Q(\mathbf{r},t) Q(\mathbf{r}',t') \\ &\quad \times \delta''\left(t' - t + \frac{\mathbf{n}}{c}\cdot(\mathbf{r}-\mathbf{r}')\right) d\mathbf{r}\, d\mathbf{r}'\, dt\, dt' \\ &= \int_{-\infty}^{\infty} E(\mathbf{n},\omega)\, d\omega = \check{E}(\theta) = \int_{-\infty}^{\infty} \hat{E}(\theta,\sigma)\, d\sigma \end{aligned} \tag{A3}$$

and

$$E(\mathbf{n},\omega) = \frac{\omega^2}{2\pi\rho_0 c (4\pi)^2}\left|\int_{-\infty}^{\infty} Q(\mathbf{r},t) e^{i\omega(t-\mathbf{n}\cdot\mathbf{r}/c)}\, d\mathbf{r}\, dt\right|^2 \tag{A4}$$

describe the angular distribution and frequency distribution of the radiated energy. The particular source function

$$Q(\mathbf{r},t)=\alpha\dot{x}_s(t)\frac{\partial}{\partial x}\delta[x-x_s(t)] \tag{A5}$$

pertains to a point dipole in motion with colinear axis, as is used in Eq. (4).

APPENDIX B

Here we show the details for the simplification of the integrals in Eqs. (33), (34), and (35). Each of these integrals can be reduced to the sum of simpler integrals by successive integrations by parts. Intermediate results are

$$I_1=\frac{4}{21}\frac{d^2}{dM^2}\left[\int_{-\pi/2}^{\pi/2}\left(\frac{1}{D^6}-\frac{1}{D^5}+\frac{3}{16D^4}\right)d\sigma\right], \tag{B1}$$

where

$$D=1-M^2\sin^2\sigma, \tag{B2}$$

and

$$I_2/(2M)=-\frac{1}{210}\frac{d^2}{dM^2}\left[\left(3M-\frac{2}{M}\right)\int_0^{\pi/2}\left(\frac{80}{D^6}-\frac{48}{D^5}+\frac{3}{D^4}\right)\times d\sigma+\frac{2}{M}\int_0^{\pi/2}\left(\frac{80}{D^5}-\frac{48}{D^4}+\frac{3}{D^3}\right)d\sigma\right], \tag{B3}$$

$$I_3/M^2=\frac{4}{35}\frac{d}{dM}\left\{M\left[63\int_0^{\pi/2}\frac{d\sigma}{D^7}+(234M^2-84)\times\int_0^{\pi/2}\frac{\sin^2\sigma\,d\sigma}{D^7}+(63M^4-312M^2+28)\times\int_0^{\pi/2}\frac{\sin^4\sigma\,d\sigma}{D^7}+(104M^2-84M^4)\times\int_0^{\pi/2}\frac{\sin^6\sigma\,d\sigma}{D^7}+28M^4\int_0^{\pi/2}\frac{\sin^8\sigma\,d\sigma}{D^7}\right]\right\}. \tag{B4}$$

The above integrals all admit representations in terms of hypergeometric functions[13] of argument M^2. The pertinent relation is[14]

$$H_{a,n}(M^2)=\int_0^{\pi/2}\frac{\sin^{2n}\sigma\,d\sigma}{(1-M^2\sin^2\sigma)^a}=\frac{\Gamma(n+\frac{1}{2})\sqrt{\pi}}{2\Gamma(n+1)}F(a,n+1/2;n+1;M^2), \tag{B5}$$

where F is the hypergeometric function, and Γ is the gamma function. The above integrals can thus be reduced to the form

$$I_1=\frac{8}{21}\frac{d^2}{dM^2}\left(H_{6,0}-H_{5,0}+\frac{3}{16}H_{4,0}\right), \tag{B6}$$

$$I_2/M=-\frac{1}{210}\frac{d^2}{dM^2}\left[\left(3M-\frac{2}{M}\right)(80H_{6,0}-48H_{5,0}+3H_{4,0})+\frac{2}{M}(80H_{5,0}-48H_{4,0}+3H_{3,0})\right], \tag{B7}$$

$$I_3/M^2=\frac{4}{35}\frac{d}{dM}\{M[63H_{7,0}+(234M^2-84)H_{7,1}+(63M^4-312M^2+28)H_{7,2}+(104M^2-84M^4)H_{7,3}+28M^4H_{7,4}]\}. \tag{B8}$$

This is a convenient form in which to express the solution, since, using the hypergeometric series[13]

$$F(a,n+1/2;n+1;M^2)=1+\frac{a(n+\frac{1}{2})}{1!(n+1)}M^2+\frac{a(a+1)(n+\frac{3}{2})(n+\frac{1}{2})}{2!(n+1)(n+2)}M^4+\cdots, \tag{B9}$$

all the integrals can be expressed in terms of even powers of the Mach number M. Therefore,

$$I_1=\frac{\pi}{3}\left(1+\frac{81}{4}M^2+\cdots\right), \tag{B10}$$

$$I_2=\frac{\pi}{3}\left(1+\frac{27}{5}M^2+\cdots\right), \tag{B11}$$

$$I_3=\frac{\pi}{3}\left(\frac{27}{5}M^2+\frac{675}{14}M^4+\cdots\right). \tag{B12}$$

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