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Elasticity solution for an axially compressed and encased cylindrical specimen

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Abstract

We present the solution of the linear elasticity equations governing the deformation of an elastic cylinder encased in a tube and subjected to uniform compression on the flat ends. The solutions for the stresses, strains, and displacements in the encased and compressed cylinder are all systematically determined from the basic solution of Lamé's classical elasticity problem of the long tube subjected to internal and external pressures. We first derive the general elastostatic analysis for an encased hollow cylinder, stress-free at the cavity, and later particularize the solution to a solid cylindrical specimen. The effective modulus E_{eff} of the encased sample is found to be a function of the bulk modulus k and Poisson's ratio ν of the material. E_{eff} differs from k except for nearly incompressible materials, where E_{eff} approaches the bulk modulus value. In the incompressible case, we also show how a load applied on the cylinder's flat ends is equivalent to, and can be replaced by, the same load acting on the curved surface. For compressible materials, a more general expression for E_{eff} is found that also accounts for the case deformation. These results explain the deformation of an axially compressed and encased cylindrical specimen tested in compressibility measuring devices such as those described by Matsuoka and Maxwell [Response of linear high polymers to hydrostatic pressure. Journal of Polymer Science 1958; 32:131–59]. The present analysis thus contributes to a better understanding of how this device works and to the interpretation of measurements taken with it.

Keywords: Elasticity; Encased cylinder; Compressibility

1. Introduction

The determination of the compressibility of a polymeric material sample is an important problem that has attracted much attention in the past. There have been comprehensive studies on the subject [1, Chapter 6] as well as a number of papers [2,3] that describe various testing devices in common use to carry out such determination. Perhaps the best known such device is the one described by Matsuoka and Maxwell [4]. Warfield [3] claimed that measurements obtained with this device yielded the bulk modulus of the sample under investigation. However, no detailed theoretical analysis has yet appeared studying the elastic deformation process taking place in a cylindrical sample encased in a tube and subjected to compression on the flat ends in a way typical of these material testers.

In order to produce such a model solution, we present here a rather general three-dimensional elastostatic problem with axial symmetry, and we reduce the general solution to some special cases of interest. The complete solution will determine the stresses, strains, and displacements everywhere in the cylindrical sample under test and also yield the sample's effective modulus E_{eff}, which is the quantity really determined by measurements made using the tester. As we will see, E_{eff} is not, in general, the bulk modulus of the sample material. Thus, the purpose of this analysis is to provide a theoretical understanding of the workings of these devices, particularly the one described by Matsuoka and Maxwell [4]. This understanding will be

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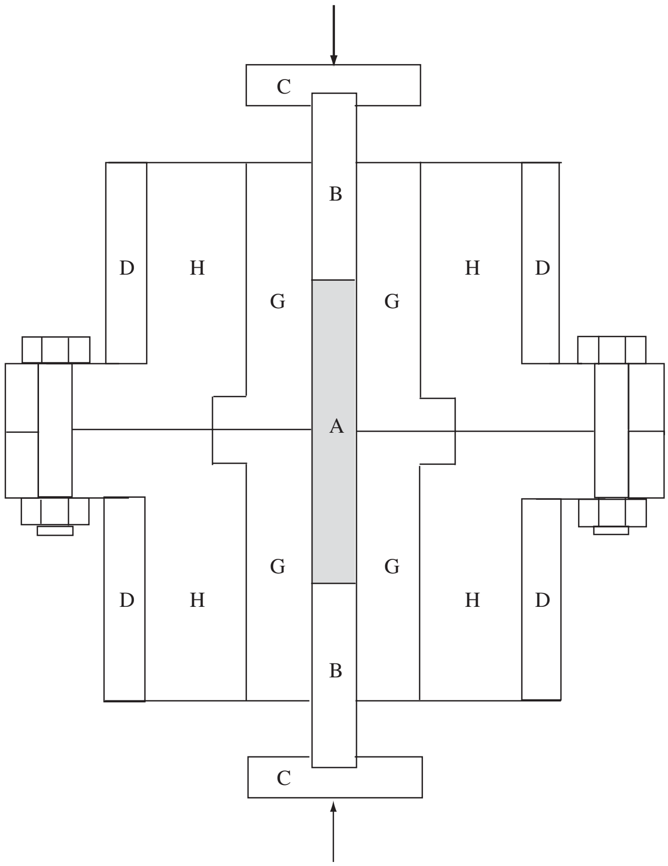

Fig. 1. Typical hydrostatic pressure application device.

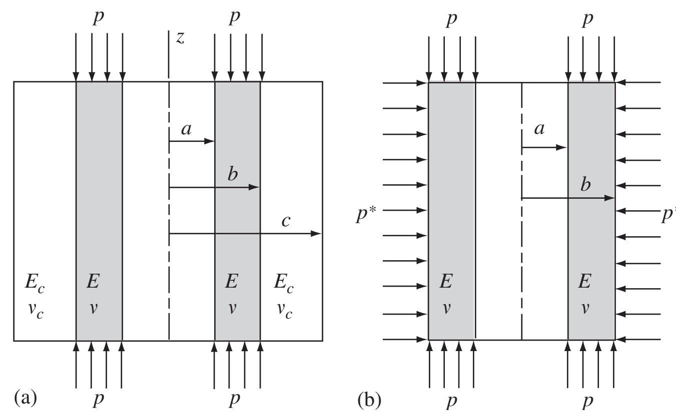

Fig. 2. Geometry and loads on an elastic hollow cylinder encased in an elastic tube and subjected to uniform compression on the flat ends: (a) composite system, (b) cylinder with interface load p^*.

helpful in analyzing whatever data may be taken with the device. The compressibility of a material sample has applications in many fields, including the design of underwater sound absorbers [5], which are rubber coatings applied on the surface of underwater structures to reduce the echoes produced when ensonified by searching sonars. Knowledge of the material properties of these materials is essential for their proper design. Much effort has been devoted over the years to the measurement of their absorptive properties [6,7].

The description of the Matsuoka–Maxwell tester [4] is quite straightforward and illustrated in Fig. 1.

The cylindrical specimen is placed in the cavity (A). The compressing plungers (B), which are made of nitralloy steel, fit the cavity quite closely so that an almost air-tight, smooth-moving fit is obtained. A cylindrical bushing (G) made of hardened steel is fitted inside a steel casing (H). The bases of the plunger (C) are mounted on the platens of a universal testing machine that applies hydrostatic pressure on the specimen through the plungers. The testing machine can measure both the applied force and the resulting displacement. Electric heating bands (D) are wrapped around the cylinders for temperature control. Other details of the device, not pertinent to the present work, are given by Matsuoka and Maxwell [4].

2. Problem formulation and solution

We will assume that the cylindrical specimen (A) in Fig. 1 is long enough compared to its diameter that we can treat the associated stress problem as a two-dimensional elasticity problem. Thus, in accordance with Saint-Venant's principle, a two-dimensional solution will be valid over most of the length of the specimen but not near the ends, where there will be some end effects. We also neglect any friction effects in the operation of the device.

Consider a hollow cylinder of finite length with inner and outer radii a and b, respectively (Fig. 2).

The cylinder material has elastic modulus E and Poisson's ratio v. The cylinder is encased in an outer tube of inner and outer radii b and c, respectively, and material constants E_c and v_c. The encased cylinder is subjected to a uniform compressive pressure p applied at the top and bottom flat surfaces. This pressure is the only load applied to the composite system. The goal is to determine the stresses, strains, and displacements everywhere in the structure.

The basis for the elasticity solution is the classical Lamé solution [8] for a tube subjected to both internal and external pressures. This solution is available in various references on elasticity [9,10]. The key result is that, for a hollow cylindrical tube of inside radius a and outside radius b subjected to uniform inside and outside pressures p_i and p_o, respectively, the stresses are given by

$$\left\{ \begin{matrix} \sigma_r \\ \sigma_\theta \end{matrix} \right\} = \frac{p_i a^2}{b^2 - a^2}\left(1 \mp \frac{b^2}{r^2}\right) - \frac{p_o b^2}{b^2 - a^2}\left(1 \mp \frac{a^2}{r^2}\right). \tag{1}$$

Since the sum $\sigma_r + \sigma_\theta$ is constant, these stresses produce a uniform extension or contraction in the longitudinal direction, and cross-sections perpendicular to the axis remain plane. The usefulness of this solution is that it can be applied to both the cylindrical specimen and the elastic case.

2.1. Solution for the cylinder

When a pressure p is applied to the ends of the cylindrical specimen, the cylinder expands laterally and applies an interface pressure, denoted p^* (positive in compression), to the case. The reaction on the cylinder is shown in Fig. 2. Thus, from Eq. (1), the stresses in the cylinder are

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = -\frac{p^* b^2}{b^2 - a^2}\left(1 \mp \frac{a^2}{r^2}\right), \quad \sigma_z = -p. \tag{2}$$

The corresponding strains are obtained from Hooke's law as

$$\begin{aligned} \varepsilon_r &= [\sigma_r - \nu(\sigma_\theta + \sigma_z)]/E, \\ \varepsilon_\theta &= [\sigma_\theta - \nu(\sigma_r + \sigma_z)]/E, \\ \varepsilon_z &= [\sigma_z - \nu(\sigma_r + \sigma_\theta)]/E. \end{aligned} \tag{3}$$

Thus, for the cylinder,

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{Bmatrix} = \frac{1}{E}\left\{ \nu p - \frac{p^* b^2}{b^2 - a^2}\left[(1-\nu) \mp (1+\nu)\frac{a^2}{r^2}\right]\right\}, \tag{4}$$

$$p^* = \frac{\nu p}{\frac{b^2}{b^2-a^2}[(1-\nu)+(1+\nu)a^2/b^2] + \frac{b^2}{c^2-b^2}[(1-\nu_c)+(1+\nu_c)c^2/b^2]E/E_c}. \tag{14}$$

$$\varepsilon_z = -\frac{1}{E}\left(p - \frac{2\nu p^* b^2}{b^2 - a^2}\right), \tag{5}$$

and the displacements are given by

$$u = r\varepsilon_\theta = \frac{r}{E}\left\{ \nu p - \frac{p^* b^2}{b^2 - a^2}\left[(1-\nu) + (1+\nu)\frac{a^2}{r^2}\right]\right\}, \tag{6}$$

$$w = z\varepsilon_z = -\frac{z}{E}\left(p - \frac{2\nu p^* b^2}{b^2 - a^2}\right). \tag{7}$$

2.2. Solution for the case

We use primes to denote the stresses, strains, and displacements in the case to distinguish them from the solution variables in the cylinder, which are unprimed. The inner cylinder pushes the case out with pressure p^*. The outer case is a thick-walled cylinder under internal pressure p^*. Thus, from Eq. (1),

$$\begin{Bmatrix} \sigma'_r \\ \sigma'_\theta \end{Bmatrix} = \frac{p^* b^2}{c^2 - b^2}\left(1 \mp \frac{c^2}{r^2}\right), \quad \sigma'_z = 0. \tag{8}$$

The last equation follows since the case is unloaded axially.

The corresponding strains are obtained from Hooke's law as

$$\begin{aligned} \varepsilon'_r &= [\sigma'_r - \nu_c(\sigma'_\theta + \sigma'_z)]/E_c, \\ \varepsilon'_\theta &= [\sigma'_\theta - \nu_c(\sigma'_r + \sigma'_z)]/E_c, \\ \varepsilon'_z &= [\sigma'_z - \nu_c(\sigma'_r + \sigma'_\theta)]/E_c. \end{aligned} \tag{9}$$

Thus, for the case,

$$\begin{Bmatrix} \varepsilon'_r \\ \varepsilon'_\theta \end{Bmatrix} = \frac{p^* b^2}{E_c(c^2 - b^2)}\left[(1-\nu_c) \mp (1+\nu_c)\frac{c^2}{r^2}\right], \tag{10}$$

$$\varepsilon'_z = -\frac{2\nu_c p^* b^2}{E_c(c^2 - b^2)}, \tag{11}$$

and the displacements are given by

$$u' = r\varepsilon'_\theta = \frac{r p^* b^2}{E_c(c^2 - b^2)}\left[(1-\nu_c) + (1+\nu_c)\frac{c^2}{r^2}\right], \tag{12}$$

$$w' = z\varepsilon'_z = -\frac{2\nu_c z p^* b^2}{E_c(c^2 - b^2)}. \tag{13}$$

2.3. Determination of p^*

The solutions given above for the cylinder and case are expressed in terms of the actual load p applied on the flat ends of the cylinder and the unknown interface pressure p^*. To determine p^*, we match the radial displacements for the cylinder and case at the interface; i.e., we set $u = u'$ at $r = b$. This condition yields Eq. (14).

The complete solution of this problem for the stresses, strains, and displacements in the cylinder and case is therefore given by Eqs. (2)–(13), with p^* given by Eq. (14). It is not necessary at this point to show the results of this substitution, since we are primarily interested in the two limiting situations to be considered next. However, we will return to this general solution toward the end of the Discussion section.

2.4. Rigid case

If the case is rigid ($E_c \to \infty$), with other parameters unchanged, the strains and displacements in the case vanish, and the interface pressure p^* reduces to

$$p^* = \frac{b^2 - a^2}{b^2}\frac{\nu p}{(1-\nu) + (1+\nu)a^2/b^2}. \tag{15}$$

The stresses in the case are, from Eq. (8),

$$\begin{Bmatrix} \sigma'_r \\ \sigma'_\theta \end{Bmatrix} = \frac{b^2 - a^2}{c^2 - b^2}\left(1 \mp \frac{c^2}{r^2}\right)\frac{\nu p}{(1-\nu) + (1+\nu)a^2/b^2}, \quad \sigma'_z = 0. \tag{16}$$

The stresses, strains, and displacements in the cylinder are

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \end{Bmatrix} = -\left(1 \mp \frac{a^2}{r^2}\right)\frac{\nu p}{(1-\nu) + (1+\nu)a^2/b^2}, \quad \sigma_z = -p, \tag{17}$$

$$\left\{ \begin{array}{c} \varepsilon_r \\ \varepsilon_\theta \end{array} \right\} = \frac{vp}{E} \left[1 - \frac{(1-v) \mp (1+v)a^2/r^2}{(1-v)+(1+v)a^2/b^2} \right], \tag{18}$$

$$\varepsilon_z = -\frac{p}{E} \frac{(1+v)(1-2v)+(1+v)a^2/b^2}{(1-v)+(1+v)a^2/b^2}, \tag{19}$$

$$u = \frac{rpv}{E} \left[1 - \frac{(1-v)+(1+v)a^2/r^2}{(1-v)+(1+v)a^2/b^2} \right], \tag{20}$$

$$w = -\frac{zp}{E} \frac{(1+v)(1-2v)+(1+v)a^2/b^2}{(1-v)+(1+v)a^2/b^2}. \tag{21}$$

We observe from the above solution that the boundaries $r = a$ (on the cylinder) and $r = c$ (on the case) are both stress-free, and, at the interface $r = b$, the radial displacement u and hoop strain ε_θ both vanish, as required.

Under the present approximation of a rigid case, we can define an effective Young's modulus E_{eff} given by

$$E_{\text{eff}} = \frac{\sigma_z}{\varepsilon_z}. \tag{22}$$

The substitution of Eqs. (17) and (19) into this equation yields

$$E_{\text{eff}} = E \frac{(1-v)+(1+v)a^2/b^2}{(1+v)(1-2v)+(1+v)a^2/b^2}. \tag{23}$$

If the cylinder has no hole ($a = 0$), then, from Eq. (15),

$$p^* = \frac{v}{1-v} p, \tag{24}$$

and, from Eq. (23),

$$E_{\text{eff}} = E \frac{1-v}{(1+v)(1-2v)} = \lambda + 2\mu, \tag{25}$$

where λ and μ are the Lamé constants of elasticity [11].

If the cylinder wall thickness is small ($a \approx b$), Eq. (23) implies

$$E_{\text{eff}} \approx \frac{E}{1-v^2}, \tag{26}$$

which is the apparent Young's modulus in plane strain [11,12]. Hence, as required, the limiting values of the effective modulus agree with the known results for plane stress and plane strain.

2.5. Cylinder and case of same material

If the cylinder and case materials are the same, $E = E_c$ and $v = v_c$, in which case, from Eq. (14), the interface pressure p^* simplifies to

$$p^* = \frac{vp}{2} \frac{(1-a^2/b^2)(1-b^2/c^2)}{1-a^2/c^2}. \tag{27}$$

If the cylinder also has no hole ($a = 0$), this expression further simplifies to

$$p^* = \frac{vp}{2} (1-b^2/c^2). \tag{28}$$

The axial strain in the cylinder is found by substituting Eq. (27) into Eq. (5):

$$\varepsilon_z = -\frac{p}{E} \left(1 - v^2 \frac{1-b^2/c^2}{1-a^2/c^2} \right), \tag{29}$$

so that the corresponding effective modulus is

$$E_{\text{eff}} = \frac{\sigma_z}{\varepsilon_z} = \frac{E}{1-v^2(c^2-b^2)/(c^2-a^2)}. \tag{30}$$

If there is no case ($b = c$), $p^* = 0$, and the general solutions for cylinder and case reduce to just those of the cylinder:

$$\sigma_r = \sigma_\theta = 0, \quad \sigma_z = -p, \quad \varepsilon_r = \varepsilon_\theta = \frac{vp}{E},$$
$$\varepsilon_z = -\frac{p}{E}, \quad u = \frac{rpv}{E}, \quad w = -\frac{pz}{E}, \tag{31}$$

which corresponds to the elementary case of uniaxial compression in the z-direction. Thus, in this case, $E_{\text{eff}} = E$.

If the cylinder is absent ($a = b$),

$$E_{\text{eff}} = \frac{E}{1-v^2}, \tag{32}$$

which is the apparent Young's modulus in plane strain.

3. Discussion

We consider here several variations of the basic solution presented in the previous section.

Consider first a short solid cylinder ($a = 0$) of radius b subjected to a uniform compression p on its flat top and bottom surfaces. The stresses, strains, and displacements in the cylinder are those of uniaxial compression given in Eq. (31). There are no stress concentrations anywhere. If the pressure p acting on the flat surfaces were removed and made to act instead as a tension on the surface $r = b$ of the cylinder, a new stress situation quite different would result. This new stress field can be deduced from Eqs. (2)–(7) with the substitutions $a = 0$, $p = 0$, and $p^* = -p$:

$$\sigma_r = \sigma_\theta = p, \quad \sigma_z = 0, \tag{33}$$

$$\varepsilon_r = \varepsilon_\theta = p(1-v)/E, \quad \varepsilon_z = -2vp/E,$$
$$u = pr(1-v)/E, \quad w = -2pvz/E. \tag{34}$$

Notice that, for incompressible materials ($v = 0.5$) such as rubber, the strain and displacement fields produced by these two different stress situations in Eqs. (31) and (34) are identical:

$$\varepsilon_r = \varepsilon_\theta = p/(2E), \quad \varepsilon_z = -p/E,$$
$$u = pr/(2E), \quad w = -pz/E, \tag{35}$$

which is an interesting result.

Consider now the case of the cylinder encased in an outer rigid tube. The stresses, strains, and displacements in a solid cylindrical specimen encased in a rigid tube of the same radius can be obtained from Eqs. (17)–(21) with $a = 0$. This is the situation that can be used to model the deformation of specimens tested in the various devices used for this purpose including the Matsuoka–Maxwell device

[4]. The case is considerably stiffer and more massive than the tested specimen, so that the assumption $E_c = \infty$ is a reasonable assumption. The elasticity solution can then be written as

$$\sigma_r = \sigma_\theta = -\frac{v}{1-v}p, \quad \sigma_z = -p, \tag{36}$$

$$\varepsilon_r = \varepsilon_\theta = 0, \quad \varepsilon_z = -p(1+v)(1-2v)/[E(1-v)], \tag{37}$$

$$u = 0, \quad w = -pz(1+v)(1-2v)/[E(1-v)], \tag{38}$$

and the effective modulus E_{eff} is given by Eq. (25). The present situation is quite different from the state of uniaxial compression addressed when the two materials are the same, since now the outer rim of the cylindrical specimen is constrained. However, if the outer rim is unconstrained, E_{eff} is merely the Young's modulus E. That situation occurs before the cylindrical sample under compression has had a chance to fill completely the tester cavity, which means that the cylinder's outer rim has not yet been restrained by the case. Since the bulk modulus for an isotropic material is related to Young's modulus and Poisson's ratio by

$$k = \frac{E}{3(1-2v)}, \tag{39}$$

Eq. (25) can be written in terms of k and v as

$$E_{\mathrm{eff}} = 3k\left(\frac{1-v}{1+v}\right), \tag{40}$$

which is the effective modulus measured by the tester. For nearly incompressible materials ($v \approx 0.5$), $E_{\mathrm{eff}} \approx k$. For any other material, we should return to Eq. (40), which is plotted in Fig. 3.

Even for nearly incompressible materials, the error is significant. For example, for $v = 0.49$, $E_{\mathrm{eff}}/k = 1.03$; for $v = 0.45$, $E_{\mathrm{eff}}/k = 1.14$; and for $v = 0.40$, $E_{\mathrm{eff}}/k = 1.29$.

Consider now the situation where the deformation of the case is not negligible. The effective modulus, defined in Eq. (22), can be obtained by substituting Eqs. (2), (5), and (14) into Eq. (22). The result can be expressed in the form

$$E_{\mathrm{eff}} = E\left[\frac{1-v+A}{(1+v)(1-2v)+A}\right], \tag{41}$$

where

$$A = (1+v)\frac{a^2}{b^2} + \frac{E}{E_c}\frac{b^2-a^2}{c^2-b^2}\left[(1-v_c)+(1+v_c)\frac{c^2}{b^2}\right] \quad (b \neq c). \tag{42}$$

A is a dimensionless parameter that depends on the material properties and geometry of both the cylinder and case. If the cylindrical specimen is solid ($a = 0$),

$$A = \frac{k}{k_c}\frac{1-2v}{1-2v_c}\left(v_c + \frac{c^2+b^2}{c^2-b^2}\right), \tag{43}$$

where A is expressed in terms of the bulk moduli, Poisson's ratios, and the case dimensions.

It is also interesting to quantify the effects of taking account of the elasticity of the casing. We define the relative effect e as

$$e = (E_{\mathrm{eff(elastic)}} - E_{\mathrm{eff(rigid)}})/E_{\mathrm{eff(rigid)}}, \tag{44}$$

where, for a solid specimen ($a = 0$), $E_{\mathrm{eff(elastic)}}$ is given by Eqs. (41) and (43), and $E_{\mathrm{eff(rigid)}}$ is given by Eqs. (40) and (39). The substitution of these equations into Eq. (44) yields

$$e = -\frac{2v^2A}{(1-v)(1-v-2v^2+A)}, \tag{45}$$

where, from Eq. (43),

$$A = \frac{E}{E_c}\left(v_c + \frac{c^2/b^2+1}{c^2/b^2-1}\right). \tag{46}$$

For example, if $c/b = 2$ (a reasonable possibility),

$$A = \frac{E}{E_c}\left(v_c + \frac{5}{3}\right), \tag{47}$$

in which case

$$e = -\frac{2v^2(v_c+5/3)E/E_c}{(1-v)[1-v-2v^2+(v_c+5/3)E/E_c]}. \tag{48}$$

For hard rubber encased in steel ($E_c/E = 200$, $v = 0.2$, $v_c = 0.3$), $e = -1.3 \times 10^{-3}$. Clearly the steel case can reasonably be assumed as rigid in this situation. For softer elastomers, e would be even smaller.

Thus, in most practical cases, the deformation of the case is negligible compared to that of the specimen (similar to the rigid case situation), and then the measured bulk modulus is slightly lower than the true value. However, in the rare instances where the case deformation is significant, the measured bulk modulus is larger than the true value. The parameter A is a measure of the case deformation. Eq. (41) quantitatively establishes the resulting departure from the true values.

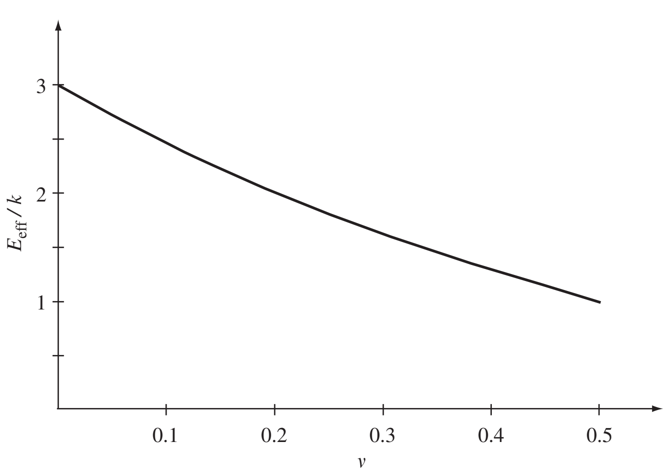

Fig. 3. Plot of E_{eff}/k vs. Poisson's ratio v (Eq. (40)).

4. Conclusions

With the Matsuoka–Maxwell and similar testing devices, the interpretation of the effective modulus as the bulk modulus is strictly valid only for incompressible materials. For other materials, these testers measure an "effective modulus" which depends on the bulk modulus and Poisson's ratio in the form given in Eq. (40).

This analysis, which is based on linear elasticity, explains the deformation of an axially compressed cylindrical specimen encased in a rigid or elastic tube. The analysis, which describes quantitatively how the tester works, will be of use in the interpretation of measurements taken with it. We have also completely solved several generalizations of the elastostatic problem for some additional situations of interest.

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